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Mark Linden Kiefer
Iowa State University

Kiefer, Mark Linden

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Nonstrange baryon properties from one-gluon exchange and linear confinement between quarks

by

Mark Linden Kiefer

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TABLE OF CONTENTS

I. INTRODUCTION

The proton and neutron, constituent particles of nuclei, are not simple, point particles. They appear to have an internal structure. For example, the neutron, a particle with zero charge, has a mean square charge radius of $-0.113\ F^2$ (1). This would be zero if the neutron had no structure. Also, if they were point particles, the proton and neutron would have, in units of nuclear magnetons, magnetic moments equal to 1 and 0, respectively. However, the proton's value is about 2.79 and the neutron's -1.91. This suggests that the proton and neutron consist of some set of circulating, charged particles. These particles are taken to be quarks.

The proton and neutron, denoted as p and n, belong to a larger class of particles known as baryons. Baryons, as will be shown in more detail, are thought to be particles consisting of three quarks or of three antiquarks. Another class of related particles is the mesons which are thought to be particles consisting of one quark and one antiquark. The mesons and baryons together make up the class of particles known as hadrons. Baryons, three quark systems, are the subject of this thesis. This chapter is a review of quark and baryon properties and much of the material is taken from the texts of Dean (2) and Kokkedee (3).

Before proceeding further, a review of the system of units to be used is needed. Planck's constant, $\hbar$, and the speed of light, c, are set equal to one. In this case, an energy unit, taken as the MeV, is the only one needed. This implies that velocity is given as a fraction

of c and angular momentum in units of $\hbar$. Since $\hbar c$ has the dimensions energy times length, then length is given in MeV^{-1}. The electromagnetic fine structure constant,

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.03604 \ldots}$$

implies that charge is given in units of the proton charge, e,

$$e = (137.03604 \ldots)^{-1/2} \quad . \tag{1.1}$$

The relation between rest mass, m, and rest energy, E_o,

$$E_o = mc^2$$

then becomes

$$E_o = m \tag{1.2}$$

and thus a particle's mass is the same as its rest energy and is given in units of MeV. The relation between total energy, E, momentum, $\underset{\sim}{p}$, and rest mass,

$$E^2 = p^2c^2 + m^2c^4$$

then becomes

$$E^2 = p^2 + m^2 \tag{1.3}$$

and thus momentum is also given in units of MeV.

To begin, some properties of quarks will be given. For the purposes of this work, only three quarks will be considered. They are the up, down, and strange quarks and are denoted as u, d, and s. In Table 1.1,

quantum numbers of these quarks are given. Q is the charge in units of the proton charge, Y is the hypercharge, τ_z is the z component of the isospin, B is the baryon number, St is the strangeness, and S is the spin.

Table 1.1. Quark quantum numbers

Quark	Q	Y	τ_z	St	S
u	+2/3	+1/3	+1/2	0	1/2
d	-1/3	+1/3	-1/2	0	1/2
s	-1/3	-2/3	0	-1	1/2

The quarks are spin 1/2 particles and so they are fermions. The u and d quarks form a $\tau = 1/2$ multiplet, τ being the total isospin. This is called an isospin doublet since the number of different states in an isospin multiplet is $2\tau+1$. The s quark has $\tau = 0$ and belongs to an isospin singlet. It will be assumed that the u and d quarks have equal mass and that the s quark is more massive than the u and d. In Chapter II, it will be shown that the u and d have masses of about 330 MeV and the s about 540 MeV. For the quantum numbers of antiquarks, just change the signs in Table 1.1, except for the spin.

The quark quantum numbers obey the two relations

$$Q = \frac{1}{2} Y + \tau_z$$

and

$$Y = St + B \quad .$$

These quantum numbers and relations come about from an underlying symmetry of the quarks and their interactions known as SU(3). This stands for the Special Unitary group of dimension three. The dimension is three because there are three types of quarks. If quarks with this SU(3) structure, or something similar, exist in baryons and mesons, then baryons may be grouped into decuplets and octets and mesons into octets and singlets. These groupings have been observed and are called supermultiplets. If the SU(3) symmetry were exact, then all the masses within a supermultiplet would be degenerate. However, this is not observed and is taken as evidence that the s quark has a mass greater than the u and d. This SU(3) symmetry of quarks, interacting with forces called the strong interactions, was proposed by Gell-Mann to explain the approximate symmetry seen in hadron spectroscopy.

The baryon octet and decuplet are represented by plotting the observed particles on a graph of their hypercharge versus their z component of isospin. The octet, shown in Fig. 1.1, consists of particles with total angular momentum, J, of 1/2 and parity, π, which is positive. The quark content of each particle is given below its label. The superscripts on the particle labels give the charge in units of e. The n and p contain u and d quarks coupled to $\tau = 1/2$. (See Appendix A for the coupling rules.) Then, the nearly equal mass n and p are interpreted as two different charge states of the nucleon, labelled as N, a particle with $\tau = 1/2$. Putting one s quark in the baryon octet yields the Σ and Λ particles. Σ has $\tau = 1$ and has three charge states, Σ^-, Σ^o, and Σ^+. Λ has $\tau = 0$ and has one charge state, Λ^o. The Ξ, with two s quarks, has

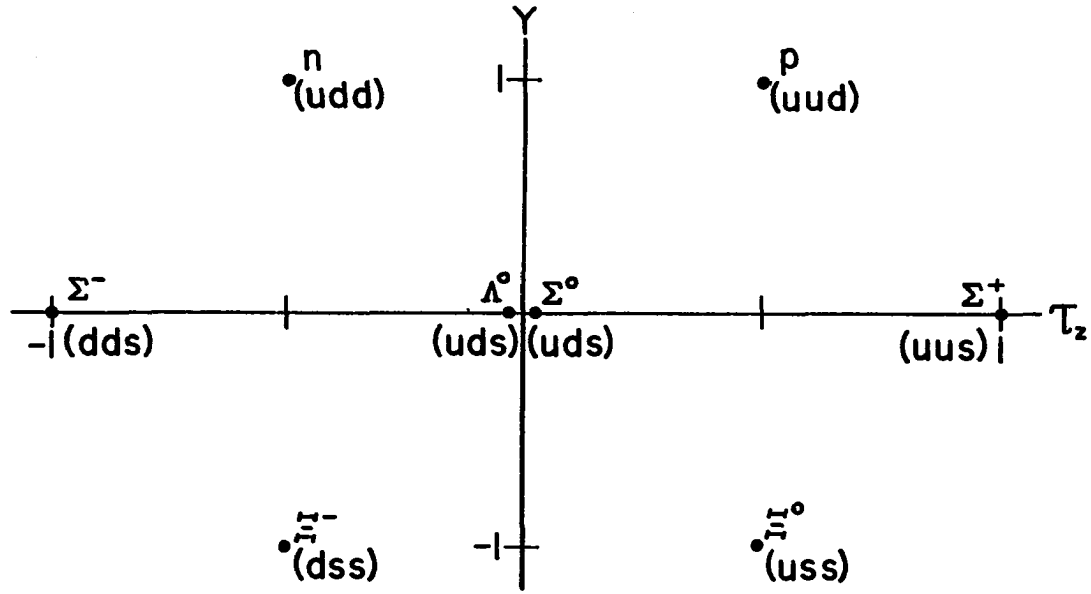

Figure 1.1. The baryon octet particles. All have $J^{\pi} = 1/2^{+}$

$\tau = 1/2$ and has two charge states, Ξ^- and Ξ^o. In Table 1.2, the masses of each particle in the baryon octet are given. These experimental values and the rest of the data in this thesis, unless so noted, are taken from the 1980 "Review of Particle Properties" (4) by the Particle Data Group. All of these particles have small mass widths of less than 0.1 MeV. The widths are small because these particles are stable against decay by strong interactions.

The isospin average is the mass assigned to the N, Σ, Λ, and Ξ particles. The differences in mass between members of the same isospin multiplet are small and may be due to electromagnetic interactions between the charged quarks and/or a small difference in the u and d masses. It will be assumed here that the u and d have equal masses and so the small, intramultiplet mass differences will then be due to electromagnetic effects. Note that as s quarks are added, the mass of the particle tends to increase. This is taken as evidence that the s quark is more massive than the u and d. The Gell-Mann--Okubo mass formula, based on an SU(3) mass operator, predicts that

$$2[N + \Xi] = \Sigma + 3\Lambda \tag{1.4}$$

where the particle label stands for its mass. This formula is satisfied to 1%. The baryon decuplet, shown in Fig. 1.2, consists of particles with $J^\pi = 3/2^+$. The Δ, similar to N in that it has no s quarks, has $\tau = 3/2$ and, thus, has four charge states, Δ^-, Δ^o, Δ^+, and Δ^{++}. The Σ^* is seen as a resonance of the octet Σ. It has $\tau = 1$ and three charge states, Σ^*, Σ^{*o}, Σ^{*+}. Similarly, the Ξ^* is a resonance of the octet Ξ.

Table 1.2. Baryon octet particle masses

Particle	Mass (MeV)	Isospin Average (MeV)	Isospin
n	940	N = 939	1/2
p	938		
Σ^-	1198		
Σ^o	1193	Σ = 1193	1
Σ^+	1189		
Λ^o	1115	Λ = 1115	0
Ξ^-	1321	Ξ = 1317	1/2
Ξ^o	1314		

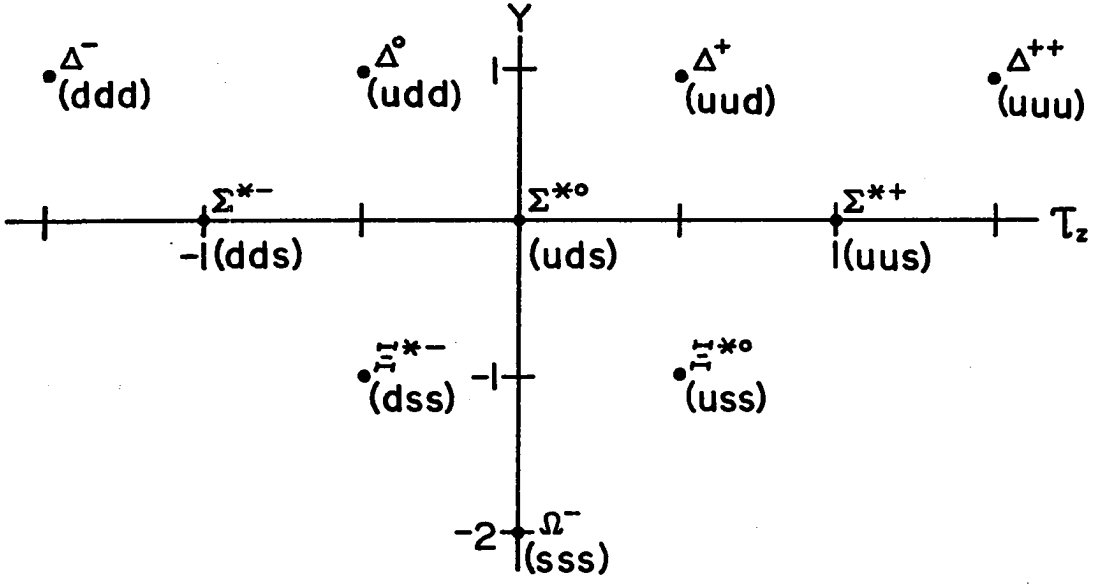

Figure 1.2. The baryon decuplet particles. All have $J^{\pi} = 3/2^{+}$

It has $\tau = 1/2$ and two charge states, Ξ^{*-} and Ξ^{*o}. The Ω^- consists of three s quarks and has $\tau = 0$. The Ω is stable against decay by strong interactions. In Table 1.3, the isospin averaged mass and the mass width of each particle in the decuplet is given.

Table 1.3. Baryon decuplet masses

Particle	Mass (MeV)	Mass Width (MeV)	Isospin
Δ	1232	115	3/2
Σ^*	1385	35	1
Ξ^*	1530	10	1/2
Ω	1672	-	0

Here, the Gell-Mann--Okubo formula predicts that

$$\Omega - \Xi^* = \Xi^* - \Sigma^* = \Sigma^* - \Delta \qquad (1.5)$$

and this is satisfied to 4%. Thus, there is an approximately equal increase in mass as one adds an s quark in the decuplet and this is taken as evidence that the s is more massive than the u and d by roughly 150 MeV. Note that the mass differences between octet and decuplet particles, such as $\Delta - N = 293$ MeV, are due to the strong interactions between quarks. Thus, the strong interactions between quarks in hadrons are roughly 100 times greater in magnitude than the electromagnetic interactions which produce the small, intramultiplet mass differences in the octet, such as $n - p \approx 2$ MeV. The N and Δ contain no strange quarks

and so are called nonstrange baryons. They are the focus of this thesis and since the u and d quarks are assumed to have equal mass, the N and Δ are taken to be systems of three equal mass particles. The Ω is also a system of three equal mass particles and a few results will also be given for it.

The N and Δ, bound states of three quarks, have resonant spectra. These resonances then occur by putting the quarks into excited states. The N spectrum below 2000 MeV is given in Table 1.4. The particle data group has rated the confidence in the experimental values of a level's mass, angular momentum, and parity on a rising scale of from one to four stars. Those in Table 1.4 are three and four star levels. There is an overall, gross structure to this spectrum. The positive parity ground state is located at 940 MeV, a shell of negative parity levels lies above that with a center of gravity at about 1600 MeV, and a positive parity shell lies above that with a center of gravity at about 1800 MeV. If quarks were bound to each other by potentials, such as a harmonic oscillator, which depend only on the distance between quarks, then the N ground state would be interpreted as one in which the three quarks are in relative 1S states. The negative parity shell would have one negative parity 1P excitation added to the system. The positive parity shell would have one positive parity 2S or 1D excitation or two 1P excitations added to the system. This structure is roughly true in the observed N spectrum. There is a fine structure to the spectra which is the mass differences within the negative and positive parity shells. This fine structure could be due to spin dependent forces between quarks, for

Table 1.4. N spectrum below 2000 MeV

Mass	J^{π}
1990	$7/2^{+}$
1810	$3/2^{+}$
1710	$1/2^{+}$
1688	$5/2^{+}$
1470	$1/2^{+}$
1700	$3/2^{-}$
1670	$5/2^{-}$
1650	$1/2^{-}$
1535	$1/2^{-}$
1520	$3/2^{-}$
939	$1/2^{+}$

example. The main goal of this thesis is to postulate a three quark Hamiltonian for the N and Δ and to compare the theoretical spectra with experiment.

This work has a direct bearing on nuclear physics in that the underlying forces between quarks can be used to derive a nuclear force. This work has been carried out by S. A. Williams, D. L. Pursey, P. D. Morley, and F. J. Margetan (5). They have taken the deuteron, a six quark system consisting of the p and n, and have used the same Hamiltonian to describe the quark forces as that used in this thesis. They have found that this Hamiltonian describes the deuteron properties well. Thus, the nuclear force is seen as originating from forces between quarks. This thesis will attempt to verify that the quark Hamiltonian used by Williams, Pursey, Morley, and Margetan is a plausible one in that it describes well the more fundamental, three quark systems.

II. A SEMIRELATIVISTIC QUARK MODEL OF BARYONS

In this chapter, a Hamiltonian describing the interactions between quarks in baryons, as well as the motivation for it, will be given. First, however, we consider some symmetry properties. Since quarks are spin 1/2 particles, we note that by adding the two quark spin states to the SU(3) symmetry of hadrons, we derive an SU(6) symmetry. If S is the total spin of the three quarks,

$$\underset{\sim}{S} = \underset{\sim}{s}_1 + \underset{\sim}{s}_2 + \underset{\sim}{s}_3 \quad , \tag{2.1}$$

then this SU(6) symmetry predicts that S = 1/2 for the octet and S = 3/2 for the decuplet. Assuming that each quark in an octet or decuplet baryon is in an identical, positive parity, 1S state, then the orbital angular momentum of the ith quark, $\underset{\sim}{\ell}_i$, is zero and, hence, the total orbital angular momentum of the three quarks,

$$\underset{\sim}{L} = \underset{\sim}{\ell}_1 + \underset{\sim}{\ell}_2 + \underset{\sim}{\ell}_3 \quad , \tag{2.2}$$

is also zero. Since the total angular momentum, J, is given as

$$\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S} \quad , \tag{2.3}$$

then J = S and, thus, the SU(6) symmetry correctly predicts the J^{π} values of the octet and decuplet. (See Appendix A for a review of angular momentum.) If each quark is in an identical 1S orbital state, then the part of the baryon wavefunction depending on the space variables, $\underset{\sim}{r}_1$, $\underset{\sim}{r}_2$, and $\underset{\sim}{r}_3$, is symmetric under particle interchange. The Ω, in particular, has S = 3/2 and, thus, the spin part of the wavefunction is symmetric as is the isospin part since there are three identical $\tau = 0$

quarks. Therefore, the Ω appears to have an overall symmetric wavefunction in the combined space, spin, and isospin variables. It can be shown that the N and Δ also have symmetric wavefunctions in space, spin, and isospin. However, quarks are fermions and so the N, Δ, and Ω, each containing three identical quarks, should have overall antisymmetric wavefunctions.

The proper, Fermi statistics of quarks are recovered in the theory of strong interactions called quantum chromodynamics and denoted as QCD. QCD will provide the motivation for the quark model in this thesis. The following is a brief review of the main postulates of QCD as presented in a paper by DeRujula, Georgi, and Glashow (6).

(1) It incorporates the previously mentioned SU(6) symmetry.

(2) It adds the new set of quark variables, color, to the hadron wavefunction. Quarks also come in three colors and color is assumed to be an exact SU(3) symmetry.

(3) All physically observable states are color singlets. This would explain why, for systems of three or less quarks, only baryons (three quarks or three antiquarks) and mesons (a quark-antiquark pair) are observed. This implies that the color part of the baryon wavefunction is antisymmetric. Thus, the N, Δ, and Ω will have antisymmetric wavefunctions in the combined space, spin, isospin, and color variables, thereby recovering Fermi statistics.

(4) The strong interactions are mediated by 8 massless, neutral, spin 1, color gluons. Since gluons are massless, neutral, and spin 1 particles, they are analogous to photons. The spin 1/2 quarks are then analogous to spin 1/2 electrons, whose interactions are mediated by photons. Thus, one may think of quarks as possessing a "color charge" in the same way that electrons possess the more familiar electric charge.

(5) This model of strong interactions displays asymptotic freedom and infrared slavery. Asymptotic freedom means that at short separations, quarks interact only weakly. Conversely, infrared slavery means that quarks are strongly bound together at

large separations and implies that quarks cannot be isolated. This is the opposite of electrons which interact weakly at large separations and strongly at small separations.

DeRujula, Georgi, and Glashow use these QCD postulates as inspiration for the following assumptions about quarks in hadrons. Their work provides the main motivation behind the Hamiltonian used in this thesis.

(1) Although in fundamental QCD the u and d quarks are assumed to be massless and the s quark to have some finite mass, they use what might be called quasi-quarks. These are bound quarks which are assumed to have an effective mass of about 1/3 the N mass for the u and d, and a slightly greater mass for the s. This means that the SU(3) symmetry discussed in Chapter I is not exact, as suggested by experiment. (See Eq. (1.5).)

(2) These quasi-quarks are assumed to be simple, Dirac, point particles and so the intrinsic magnetic moment of the ith quark is $eQ_i/2m_i$ where Q_i is the charge and m_i is the effective mass.

(3) To the extent that the u, d, and s are degenerate in mass, the overall baryon wavefunctions for the decuplet and octet particles are antisymmetric in the combined space, spin, isospin, and color variables. This implies that the wavefunctions are symmetric in the combined space, spin, and isospin variables.

(4) The property of infrared slavery assumed by the model implies that the principal binding forces between quarks in hadrons are long range forces assumed to depend only on the distance between quarks. This force does not allow a single quark to be liberated from hadrons. Note that if the masses of the u, d, and s were equal, $m_u = m_d = m_s$, and if this were the only force present between quarks, all octet and decuplet particles would be degenerate in mass. This long range force is responsible for the gross structure present in the N spectrum of Table 1.4. This is seen explicitly, for example, in the large splitting of the $1/2^+$ states, 1710 - 939 = 771 MeV.

(5) Asymptotic freedom implies that there exist weaker, short range forces between quarks that depend on the quark spins and masses. These forces are assumed to arise as if from one-gluon exchange. Since gluons are analogous to photons and quarks to electrons, these forces should have the same form as the Fermi-Breit electromagnetic interaction between electrons. This interaction is taken to order $1/c^2$ (7,8), which includes

the lowest order, nonrelativistic term plus the first order, relativistic correction. All higher order terms are assumed to be small enough that they may be ignored. This is why one calls this a semirelativistic model. In this expansion to $1/c^2$, one-gluon exchange will involve only local potentials, similar to conventional nuclear physics. There is no explicit retardation due to relativistic quark velocities and so this is sometimes called a potential model. These weaker, short range forces produce the fine structure in the N spectrum of Table 1.4. This is seen, for example, in the small splitting of the $1/2^-$ states, 1650 - 1535 = 115 MeV.

DeRujula, Georgi, and Glashow give the quark Hamiltonian for baryons and mesons as

$$H = L(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots) + \sum_i \left(m_i + \frac{p_i^2}{2m_i}\right) + \sum_{i>j} (\alpha Q_i Q_j + K\alpha_s) V_{ij} \quad , \quad (2.4)$$

where L is the long range confinement potential, and $\underset{\sim}{r}_i$, m_i, $\underset{\sim}{p}_i$, and Q_i are the position, mass, momentum, and charge of the ith quark. V_{ij} is the two body, basic 1/r-interaction expressed in the Fermi-Breit form,

$$V_{ij} = \frac{1}{r} + (\text{Darwin}) + (\text{Fermi-contact}) + (\text{spin-orbit}) + (\text{tensor-force}) \quad , \quad (2.5)$$

where

$$\text{Darwin} = -\frac{1}{2m_i m_j}\left(\frac{\underset{\sim}{p}_i \cdot \underset{\sim}{p}_j}{r} + \frac{\underset{\sim}{r} \cdot (\underset{\sim}{r} \cdot \underset{\sim}{p}_i)\underset{\sim}{p}_j}{r^3}\right) \quad , \quad (2.6)$$

$$\text{Fermi-contact} = -\frac{\pi}{2}\delta^3(\underset{\sim}{r})\left(\frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{16\underset{\sim}{s}_i \cdot \underset{\sim}{s}_j}{3m_i m_j}\right) \quad , \quad (2.7)$$

$$\text{spin-orbit} = -\frac{1}{2r^3}\left(\frac{\underset{\sim}{r}\times\underset{\sim}{p}_i\cdot\underset{\sim}{s}_i}{m_i^2} - \frac{\underset{\sim}{r}\times\underset{\sim}{p}_j\cdot\underset{\sim}{s}_j}{m_j^2} + \frac{2\underset{\sim}{r}\times\underset{\sim}{p}_i\cdot\underset{\sim}{s}_j}{m_i m_j} - \frac{2\underset{\sim}{r}\times\underset{\sim}{p}_j\cdot\underset{\sim}{s}_i}{m_i m_j}\right) , \tag{2.8}$$

and

$$\text{tensor-force} = -\frac{1}{m_i m_j r^3}\left(\frac{3(\underset{\sim}{s}_i\cdot\underset{\sim}{r})(\underset{\sim}{s}_j\cdot\underset{\sim}{r})}{r^2} - \underset{\sim}{s}_i\cdot\underset{\sim}{s}_j\right) . \tag{2.9}$$

Here, $\underset{\sim}{r} = \underset{\sim}{r}_i - \underset{\sim}{r}_j$ and $\underset{\sim}{s}_i$ is the spin of the ith quark. 1/r is the non-relativistic term and the rest is the first order, relativistic correction. The $K\alpha_S V_{ij}$ term is the one-gluon exchange potential where K is -4/3 for the quark-antiquark pair in mesons and -2/3 for the quark-quark pair in baryons. K is derived from operating on the color part of the hadron wavefunction with operators in the color variables. α_S is the strong interaction structure constant and is presumed to be positive, which means that $K\alpha_S V_{ij}$ is an attractive potential. Also, α_S is expected to be of order one. $\alpha Q_i Q_j V_{ij}$ is the one-photon exchange potential between the charged quarks and is small compared to $K\alpha_S V_{ij}$, since $\alpha \approx 1/137$. Thus, reiterating the idea presented in Chapter I, the electromagnetic interactions between quarks in hadrons are roughly 100 times smaller than the strong interactions.

DeRujula, Georgi, and Glashow then use a perturbation approach, parameterize matrix elements, and compare mass splittings with experiment (see Tables 1.2 and 1.3), where the baryon mass is defined as

$$M_i = \langle \Psi_i | H | \Psi_i \rangle \quad . \tag{2.10}$$

$|\Psi_i\rangle$ is the three quark wavefunction, antisymmetric in the combined space, spin, isospin, and color variables. As detailed at the beginning of this chapter, $|\Psi_i\rangle$ is assumed to have the quarks all in identical 1S states and so J = S. They first show that the ratio of the nonstrange quark mass to the strange quark mass is

$$\frac{m_u}{m_s} = 0.622. \tag{2.11}$$

Then they expand $1/m_s$ and $1/m_s^2$ to first order in $\Delta m = m_s - m_u$ which incurs about a 15% error by (2.11). Then they can reproduce the Gell-Mann--Okubo formulas of Eqs. (1.4) and (1.5), as well as the SU(6) relation

$$\Sigma^* - \Sigma = \Xi^* - \Xi \quad ,$$

which is experimentally satisfied to 5%. Thus, the Hamiltonian reproduces the splittings within the octet and decuplet which originate mainly from the difference in mass between the u (or d) and s quarks. It also reproduces the splittings between octet and decuplet particles which originate from the Fermi-contact term in the one-gluon exchange potential. They argue, in general, that the decuplet must be heavier than the octet because in an attractive Coulomb potential such as $K\alpha_s V_{ij}$, two Dirac particles with zero orbital angular momentum have a higher energy when their spins are aligned (the decuplet), than when opposite (the octet). Then, by setting the u and d masses at 336 MeV to give the

p magnetic moment the exact experimental result, they find respectable agreement with experiment for the known baryon magnetic moments. Note that $m_u = 336$ MeV implies that $m_s = 540$ MeV by (2.11). From the one-photon exchange term, they reproduce the Coleman-Glashow relation

$$\Sigma^+ - \Sigma^- = p - n + \Xi^0 - \Xi^- \quad ,$$

which is well-satisfied experimentally. They also find that the Hamiltonian works well for the mesons. Thus, DeRujula, Georgi, and Glashow's quark model of baryons, inspired (but not rigorously derived) from QCD, works well in describing the splittings between the masses of Tables 1.2 and 1.3 and in describing magnetic moments. However, resonant spectra to compare with Table 1.4 were not calculated. Also, the octet and decuplet baryons only provided tests of the Fermi-contact term in the one-gluon exchange and of the quark masses. The model needs a more detailed test, which will be done in this thesis for the N and Δ, the nonstrange baryons.

The Hamiltonian of (2.4)-(2.9) will be used in this thesis with only minor modifications. To be entirely consistent, since V_{ij} is taken to order $1/c^2$, the kinetic energy of the quarks must be of this order also. This means that the quark kinetic energy, E_k, must be given to order p^4. The result, which may be found by expanding (1.3) in a Taylor series in the variable p^2/m^2, is

$$E_k = \frac{p^2}{2m} - \frac{p^4}{8m^3} \quad , \tag{2.12}$$

where $E = m + E_k$. Also, the confining potential must be given explicitly. In order to reproduce the gross structure of the N spectrum of Table 1.4, a potential proportional to the distance between quarks is chosen,

$$L(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3) \propto \sum_{i>j} |\underset{\sim}{r}_i - \underset{\sim}{r}_j| \quad .$$

Note that if $L(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3)$ were harmonic (proportional to the square of the quark separation), the splitting between the ground state and negative parity shell would be equal to that between the negative parity shell and the positive parity shell. Since the first is larger than the second, a linear potential, which does not increase as rapidly with separation, is chosen.

The Hamiltonian for nonstrange baryons, systems of three equal mass quarks, is then

$$H = 3m + \sum_{i=1}^{3} \left[\frac{p_i^2}{2m} - \frac{p_i^4}{8m^3} \right] + \sum_{i>j} \underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j [V_{ij}^{OGE} + V_{ij}^{LC}] \quad , \quad (2.13)$$

where m is the mass of the u and d quarks. The $\underset{\sim}{\lambda}_i$ are vector, color operators, and when operating on the color part of the hadron wavefunction, $\underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j$ is -4/3 for mesons and -2/3 for baryons. Note that $\underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j$ operating on the color variables is zero if quark i and quark j are in different color singlets. This has ramifications for the deuteron problem. V_{ij}^{OGE} is the one-gluon exchange potential,

$$V_{ij}^{OGE} = \alpha_s \left[\frac{1}{r} - \frac{1}{2m^2 r} (\underset{\sim}{p}_i \cdot \underset{\sim}{p}_i + \hat{r} \cdot (\hat{r} \cdot \underset{\sim}{p}_i)\underset{\sim}{p}_j) - \frac{\pi \delta^3(r)}{m^2} (1 + \frac{8}{3} \underset{\sim}{s}_i \cdot \underset{\sim}{s}_j) \right.$$

$$- \frac{1}{2m^2r^3}\left(\underset{\sim}{r}\times\underset{\sim}{p}_i\cdot\underset{\sim}{s}_i - \underset{\sim}{r}\times\underset{\sim}{p}_j\cdot\underset{\sim}{s}_j + 2\underset{\sim}{r}\times\underset{\sim}{p}_i\cdot\underset{\sim}{s}_j - 2\underset{\sim}{r}\times\underset{\sim}{p}_j\cdot\underset{\sim}{s}_i\right)$$

$$- \frac{1}{m^2r^3}\left(3(\underset{\sim}{s}_i\cdot\hat{r})(\underset{\sim}{s}_j\cdot\hat{r}) - \underset{\sim}{s}_i\cdot\underset{\sim}{s}_j\right)] \qquad (2.14)$$

where $\hat{r} = \underset{\sim}{r}/r$. V_{ij}^{LC} is the linear confinement potential,

$$V_{ij}^{LC} = -k|\underset{\sim}{r}_i - \underset{\sim}{r}_j| \quad , \qquad (2.15)$$

where k is the strength of the potential. k is positive so that the potential is attractive. The one-photon exchange is not used since it is 100 times smaller than the one-gluon exchange. The mass of a baryon resonance, M_i, is then defined by Schrodinger's equation,

$$H|\Psi_i\rangle = M_i|\Psi_i\rangle \quad , \qquad (2.16)$$

where $|\Psi_i\rangle$ is the three quark wavefunction of the resonance. The exact form of the eigenfunctions of H, $|\Psi_i\rangle$, are not known and so H will be diagonalized in a complete set of states which are truncated at some point. The eigenvalues will give an approximate spectrum of the resonant masses for the N and Δ and the eigenvectors an approximation to the exact wavefunctions. Convergence of the eigenvalues will be investigated by using bases of increasing size. Note that diagonalizing H in a complete set of states would give the exact eigenvalues and eigenvectors of H. The eigenvectors so found will be used to calculate magnetic moments, charge radii, and photon decay amplitudes. There are four parameters in the model. m, α_s, and k are parameters of H and an inverse length scale, β, is a parameter of the space part of the wavefunctions. These

parameters will be fixed by fitting theoretical results to a small set of experimental data. It has been assumed that the first order, relativistic corrections to E_k and V_{ij}^{OGE} are needed, but that higher order terms are hopefully small enough to be ignored. The quark velocities in the resonant states will be examined to see if this argument is consistent with the theoretical results.

Previous research into the quark model of baryons has centered around two approaches. The first type involves potential models with effective mass quarks, the approach used in this thesis, as inspired by DeRujula, Georgi, and Glashow. The second type involves a model called the MIT bag model, which uses massless u and d quarks.

In potential model calculations, the most detailed results have been reported by Isgur, Karl, and Koniuk (9-13). They used one-gluon exchange, but were inconsistent in that they kept only the Fermi-contact and tensor-force terms and used only the p^2 term in the kinetic energy. They assumed harmonic confinement. For the N, Δ, Λ, and Σ, they examined the ground states of Tables 1.2 and 1.3, the first shell of negative parity states, and the next shell of positive parity states. (See Table 1.4 for these N shells.) They did not examine all levels together by using one Hamiltonian with one set of parameters, but rather stayed within a given shell and just used the Fermi-contact and tensor-force terms to determine the mass splittings on a relative scale. A simple parametrization of any neglected anharmonicity in the potentials was also used. Energy eigenvalues and eigenfunctions were determined by diagonalizing in a harmonic oscillator basis of states,

truncated at two units of energy above the ground state. Note that this basis accounts for only 60% of the known resonances for the N and Δ. They found excellent results for splittings within a shell, the average error being only 3% for the N and Δ levels. Also, good results were found for magnetic moments and reasonable values were determined for photon decay amplitudes. However, while they claimed to have a nonrelativistic model, their quark velocities are actually highly relativistic, being on the order of 0.8. Thus, one would expect terms of a higher order than $1/c^2$ would also be important. Because of these large velocities, they found the p charge radius, experimentally equal to 0.88 F (14), to be only 50% of that value. Also, they used two different sets of parameters and did not explicitly specify the complete Hamiltonian. Because of this, they cannot calculate energies and wavefunctions of resonances with larger J values. Celmaster's work (15) is the closest in spirit to this thesis. He has used (2.13) as his Hamiltonian, but, like Isgur, Karl, and Koniuk, has only derived splittings within a shell. He examined the ground state shell and the first negative parity shell for the N, Δ, Λ, and Σ by diagonalizing H in a basis of harmonic oscillator states truncated at one unit of energy above the ground state. Note that this basis accounts for only 35% of the known N and Δ resonances. Splittings and magnetic moments were respectable, but again quark velocities were too large to justify the exclusion of terms of a higher order than $1/c^2$. Gromes and Stamatescu (16,17) have added relativistic corrections to the linear confining potential and have used perturbation in the sense of (2.10). In concert with Isgur and Karl, they examined

only the splittings within the ground state shell, the first negative parity shell, and the next positive parity shell. They, however, found the quarks to have velocities consistent with the assumption that only terms to order $1/c^2$ are needed. They found reasonable results for mass splittings within the shells. Relativistic corrections to the confining potential are not used in this thesis because the potential's origin is unknown and because this may introduce long range spin and mass dependent potentials, in contradiction to DeRujula, Georgi, and Glashow's postulate that infrared slavery involves a potential which depends only on the quark separations.

In the MIT bag model (18,19), the u and d quarks are assumed to be massless and the s quark to have a mass of about 240 MeV. The hadrons are then assumed to consist of quarks confined within a bag and interactions occur via one-gluon exchange. The interaction responsible for mass splittings is similar to the Fermi-contact term of (2.7). However, resonant masses were calculated only for the ground state baryons and the first negative parity shell. The results are rather poor when compared to the potential models.

Thus, this thesis will be an improvement over previous potential model research because the work will be done in more detail, done with more care, and will attempt to be physically consistent with underlying assumptions. One Hamiltonian, using all terms to order $1/c^2$ and with one set of parameters, will be used to calculate all the known N and Δ resonances. The absolute energies will be calculated and not just splittings within one particular shell.

III. THE BASIS OF STATES USED IN THE DIAGONALIZATION OF THE HAMILTONIAN

As stated in Chapter II, an approximate spectrum of eigenvalues and their corresponding eigenvectors of the Hamiltonian of Eqs. (2.13)-(2.15) will be found by diagonalizing H in a basis consisting of a complete set of states, truncated at some point. Diagonalizing H means that the matrix A, consisting of elements

$$A_{mn} = \langle\phi_m|H|\phi_n\rangle \quad , \tag{3.1}$$

is to be diagonalized. The state $|\phi_n\rangle$ is an antisymmetric basis function for the N and Δ in the combined quark variables of space, spin, isospin, and color. In this chapter, the basis for the N and Δ will be specified.

The color part of the basis functions can be processed for all A_{mn} since each $|\phi_n\rangle$ is antisymmetric in the color variables. $|\phi_n\rangle$ may be written as $|\psi_n\rangle|C\rangle$, where $|\psi_n\rangle$ is the symmetric basis function in the combined space, spin, and isospin variables and $|C\rangle$ is the color basis function. Then, for the mass and kinetic energy terms in the Hamiltonian, which do not depend on any color operators,

$$\langle\psi_m|[3m + \sum_i (\frac{p_i^2}{2m} - \frac{p_i^4}{8m^3})]|\psi_n\rangle\langle C|C\rangle = \langle\psi_m|[3m + \sum_i (\frac{p_i^2}{2m} - \frac{p_i^4}{8m^3})]|\psi_n\rangle \quad .$$

Then, for the potential energy terms,

$$\sum_{i>j} \langle\psi_m|V_{ij}^{OGE} + V_{ij}^{LC}|\psi_n\rangle\langle C|\underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j|C\rangle = \langle\psi_m|-\frac{2}{3} \sum_{i>j} (V_{ij}^{OGE} + V_{ij}^{LC})|\psi_n\rangle \quad ,$$

since $\underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j |C\rangle = -2/3|C\rangle$. Thus, the elements of A may be written as

$$A_{mn} = \langle\psi_m|H|\psi_n\rangle \quad , \tag{3.2}$$

where $\underset{\sim}{\lambda}_i \cdot \underset{\sim}{\lambda}_j$ is replaced by -2/3 in (2.13).

Next, we consider four symmetry properties of H. These will be used to partition the basis functions into sets such that $A_{mn} = 0$ unless $|\psi_m\rangle$ and $|\psi_n\rangle$ are in the same set. Appropriate quantum numbers will be used to differentiate the sets.

The first symmetry property of H is that it is invariant to simultaneous rotations in the three quark space and spin coordinates. This will be displayed explicitly in Chapter IV. Thus, the basis functions may be chosen to be simultaneous eigenfunctions of $\underset{\sim}{J}^2$, as given by (2.1)-(2.3), and the z-component of $\underset{\sim}{J}$, J_0. Because H is invariant under rotations, it is a tensor of rank zero. If $|\psi_m\rangle = |J'M'\rangle$ and $|\psi_n\rangle = |JM\rangle$, then by (7.30),

$$\begin{aligned} A_{mn} &= \langle J'M'|H|JM\rangle \\ &= C(JOJ';\ MOM')\langle J'||H||J\rangle \\ &= \delta_{JJ'}\delta_{MM'}\langle J'||H||J\rangle \quad , \end{aligned} \tag{3.3}$$

Thus, $A_{mn} = 0$ unless $J = J'$ and $M = M'$ and, hence, the basis is partitioned into sets labeled by the J and M quantum numbers. All members of a particular set have the same J and same M values. By (3.3), A_{mn} is independent of M and so the eigenvalues of J_0 will not be specified in the diagonalization procedure.

Similarly, the second symmetry property is that H is also a scalar under rotations in the three quark isospin coordinates, since it contains no isospin operators. Thus, the basis functions may be chosen to be eigenfunctions of $\underset{\sim}{\tau}^2$ and $\underset{\sim}{\tau}_0$, where

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}_1 + \underset{\sim}{\tau}_2 + \underset{\sim}{\tau}_3 \tag{3.4}$$

and $\tau_i = 1/2$ is the isospin of the ith quark. Then, $A_{mn} = 0$ unless $|\psi_m\rangle$ and $|\psi_n\rangle$ have the same τ and the same M_τ values. Members of the same set will have equal isospins. Hence, this separates the N basis functions from the Δ basis functions. Defining τ_{12} by

$$\underset{\sim}{\tau}_{12} = \underset{\sim}{\tau}_1 + \underset{\sim}{\tau}_2 \quad , \tag{3.5}$$

then three possible isospin couplings of the three nonstrange quarks occur,

$$|[[1/2 \times 1/2]1 \times 1/2]3/2,M_\tau\rangle \quad , \quad |[[1/2 \times 1/2]1 \times 1/2]1/2,M_\tau\rangle \quad ,$$
$$\text{and} \quad |[[1/2 \times 1/2]0 \times 1/2]1/2,M_\tau\rangle \quad . \tag{3.6}$$

These are labeled as $|[[\tau_1 \times \tau_2]\tau_{12} \times \tau_3]\tau,M_\tau\rangle$. A shorthand notation, $|(\tau_{12})\tau,M_\tau\rangle$, will be used in the following. The Δ has $\tau = 3/2$; therefore, the isospin part of the Δ basis functions will be the state $|(1)3/2,M_\tau\rangle$. The N has $\tau = 1/2$ and the isospin part of the N basis functions will consist of $|(1)1/2,M_\tau\rangle$ and $|(0)1/2,M_\tau\rangle$. As with the M quantum numbers, A_{mn} is independent of M_τ and so these values are not specified in the diagonalization. All members of an isospin multiplet have the same mass in this model.

The third symmetry property of H is that it is invariant under reflection of the space coordinates,

$$P_\pi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3) = (-\underset{\sim}{r}_1,-\underset{\sim}{r}_2,-\underset{\sim}{r}_3) \quad ,$$

where P_π is the reflection operator. This implies that the basis functions may be chosen to be eigenfunctions of P_π with eigenvalues π. The parity of the basis function is given by π and a negative parity function has $\pi = -1$ and a positive parity function has $\pi = +1$. Parity is given as a superscript on the J values. If $|\psi_m\rangle = |J^{\pi'}M\rangle$ and $|\psi_n\rangle = |J^{\pi}M\rangle$, then

$$\begin{aligned} A_{mn} &= \langle J^{\pi'}M|H|J^{\pi}M\rangle \\ &= \langle J^{\pi'}M|P_\pi^\dagger[P_\pi H P_\pi^\dagger]P_\pi|J^{\pi}M\rangle \\ &= (-1)^{\pi+\pi'}\langle J^{\pi'}M|H|J^{\pi}M\rangle \\ &= (-1)^{\pi+\pi'}A_{mn} \quad , \end{aligned}$$

where $P_\pi P_\pi^\dagger = P_\pi^\dagger P_\pi = 1$. Thus, A_{mn} is zero unless $\pi = \pi'$ and this partitions the basis into sets labeled by the π values. Parity is a conserved quantity in the strong and electromagnetic interactions and total angular momentum is always conserved. Hence, invariance of H under rotations and reflections is a necessary property in any model of the strong interactions.

The fourth symmetry property of H is that it is invariant under any interchange of the three particle indices in the combined space, spin, isospin, and color variables. Thus, H has the group S_3 as a symmetry

group. (See Appendix B for a review of S_3.) This implies that the basis functions may be chosen to be S_3 irrep basis functions under simultaneous interchange of indices in the space, spin, isospin, and color variables. The irrep label, j, would then label the different sets. Physics dictates, since quarks are fermions, that only the antisymmetric irrep, $j = [1^3]$, of S_3 is allowed. As postulated in Chapter II, the color part of the wavefunction is antisymmetric. Thus, the space-spin-isospin part is symmetric and so forms the $|111\rangle$ basis function in the [3] irrep. This is shown explicitly in Table 8.4, where coupling the [3] irrep (space-spin-isospin) with the $[1^3]'$ irrep (color) yields the $[1^3]''$ irrep (space-spin-isospin-color). To form these symmetric basis functions, the rules for coupling three S_3 irreps, one each for the space, spin, and isospin parts of the basis functions, to give an overall [3] irrep, must be specified.

Because operators of space-spin rotations, isospin rotations, reflections, and permutation of indices all commute, the basis functions may be simultaneous eigenfunctions of $\underset{\sim}{J}^2$, J_0, $\underset{\sim}{\tau}^2$, τ_0, and P_π and may also be S_3 basis functions. Thus, the quantum numbers J, M, τ, M_τ, and π, and the S_3 space-spin-isospin-color irrep label, j, may be used simultaneously to partition the basis functions. Further, it follows that, if $|\psi_m\rangle$ has the values J', M', τ', M'_τ, π', and j' and if $|\psi_n\rangle$ has the values J, M, τ, M_τ, π, and j, then $A_{mn} = 0$ unless $J' = J$, $M' = M$, $\tau' = \tau$, $M'_\tau = M_\tau$, $\pi' = \pi$, and $j' = j$. Note that only the space-spin-isospin-color $[1^3]$ irrep is used and A_{mn} is independent of M and M_τ. Hence, only J, τ, and π will be used to label the sets.

In order to build the space-spin-isospin [3] irreps, the various spin and isospin wavefunctions, of which all possibilities are known, must be classified as S_3 irreps. The space part of the basis functions, depending on the variables $\underset{\sim}{r}_1$, $\underset{\sim}{r}_2$, and $\underset{\sim}{r}_3$, will be classified once a complete set of states in the space variables is chosen. By using the projection operators of Table 8.3, it may be shown that the three types of isospin wavefunctions in (3.6) are classified as in Table 3.1, where the shorthand notation is used.

Table 3.1. The N and Δ three quark isospin functions classified as S_3 basis functions

Isospin Function	S_3 Irrep	S_3 Basis Function
$\|(1)3/2,M_\tau\rangle$	[3]	$\|111\rangle$
$\|(1)1/2,M_\tau\rangle$	[2,1]	$\|211\rangle$
$\|(0)1/2,M_\tau\rangle$	[2,1]	$\|121\rangle$

Three spin 1/2 quarks are coupled, in (2.1), to give a total spin, S. This is completely analogous to the previous isospin coupling. Thus, if

$$\underset{\sim}{S}_{12} = \underset{\sim}{s}_1 + \underset{\sim}{s}_2 \quad , \tag{3.7}$$

again three possible couplings occur. Using the shorthand notation, $|[[s_1 \times s_2]S_{12} \times s_3]S,M_s\rangle = |(S_{12})S,M_s\rangle$, where $s_i = 1/2$, then the S_3 classifications of the spin functions, given in Table 3.2, are analogous to the classifications of isospin functions.

Table 3.2. The three quark spin functions classified as S_3 basis functions

Spin Function	S_3 Irrep	S_3 Basis Function
$\|(1)3/2,M_s\rangle$	[3]	$\|111\rangle$
$\|(1)1/2,M_s\rangle$	[2,1]	$\|211\rangle$
$\|(0)1/2,M_s\rangle$	[2,1]	$\|121\rangle$

Note that the $[1^3]$ irrep does not occur because there are only two states for the three spin (or isospin) 1/2 quarks. The z-component of quark spin may be either +1/2 or -1/2. A $[1^3]$ irrep requires at least three different states. Also, the [2,1] irrep requires at least two different states, while the [3] irrep requires at least one. This is why, for example, the isospin part of the Ω wavefunction must be the symmetric, [3] irrep. Only the $|\tau_i,M_{\tau i}\rangle = |0,0\rangle$ state is available because there are three s quarks.

If we assume that all S_3 irreps occur when the space wavefunctions are classified, then the rules for coupling the space, spin, and isospin wavefunctions to a [3] irrep may be specified for the N and Δ. The N has $\tau = 1/2$, which is the [2,1] irrep. The space-spin-isospin irrep must be [3]. Hence, from Table 8.4, the space-spin irrep must be [2,1]. If $S = 3/2$, then the spin irrep is [3] and so the space irrep must be [2,1]. If $S = 1/2$, the spin irrep is [2,1] and so the space irrep may be [3], [2,1], or $[1^3]$. The Δ has $\tau = 3/2$, which is the [3] irrep. Hence, the space-spin irrep must be [3]. If $S = 3/2$, the [3] irrep, then the space irrep must be [3]. If $S = 1/2$, the [2,1] irrep, then the space irrep

must be [2,1]. It will be shown that the space S_3 irrep basis functions are eigenfunctions of $\underset{\sim}{L}^2$, L_0, and P_π, where $\underset{\sim}{L}$ is the total orbital angular momentum of (2.2). Similarly, as shown in Table 3.2, the spin irreps are eigenfunctions of $\underset{\sim}{S}^2$ and S_0. Thus, when the S_3 coupling of the space and spin irreps is performed, the angular momentum coupling of space and spin, $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$, is also performed. Table 3.3 summarizes the six previous S_3 rules for coupling space, spin, and isospin. The S_3 and angular momentum coupling notation is $((j_{L^\pi} \times j'_S)k_{J^\pi} \times j''_\tau)[3]$, where j is the space S_3 irrep with orbital angular momentum and parity L^π, j' is the spin S_3 irrep with spin S, k is the coupled space-spin S_3 irrep with angular momentum and parity J^π, j' is the isospin S_3 irrep with isospin τ, and [3] is the coupled space-spin-isospin S_3 irrep. Following this is the explicit form of the space-spin-isospin $|111\rangle$ basis function in terms of the individual space, spin, and isospin S_3 basis functions. The space S_3 functions are unprimed, the spin primed, and the isospin double-primed. (Tables 3.1 and 3.2 give the explicit form of the spin and isospin irreps.) Rules 1-4 apply to the N, which has $\tau = 1/2$, and 5 and 6 apply to the Δ, which has $\tau = 3/2$.

Next, the set of functions in the space variables must be specified. However, because only the internal motion of the quarks is relevant to the structure of the N and Δ resonances, a coordinate system which explicitly factors out the center of mass must be used. The Jacobi coordinates, $\underset{\sim}{R}$, $\underset{\sim}{\lambda}$, and $\underset{\sim}{\rho}$, are used. These are given in Table 3.4 for three equal mass particles, along with their conjugate momenta, where $\underset{\sim}{p} = -i\underset{\sim}{\nabla}$. The inverse transformations are also given. $\underset{\sim}{R}$ is the center

Table 3.3. Coupling the S_3 irreps of space, spin, and isospin in the N and Δ to give an overall [3] irrep

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(1) $(([2,1]_{L^\pi} \times [3]'_{3/2})[2,1]_{J^\pi} \times [2,1]''_{1/2})[3]$

$$\frac{1}{\sqrt{2}}\left[(|211\rangle_{L^\pi}|111\rangle'_{3/2})_{J^\pi}|211\rangle''_{1/2} + (|121\rangle_{L^\pi}|111\rangle'_{3/2})_{J^\pi}|121\rangle''_{1/2}\right]$$

(2) $(([3]_{L^\pi} \times [2,1]'_{1/2})[2,1]_{J^\pi} \times [2,1]''_{1/2})[3]$

$$\frac{1}{\sqrt{2}}\left[(|111\rangle_{L^\pi}|211\rangle'_{1/2})_{J^\pi}|211\rangle''_{1/2} + (|111\rangle_{L^\pi}|121\rangle'_{1/2})_{J^\pi}|121\rangle''_{1/2}\right]$$

(3) $(([2,1]_{L^\pi} \times [2,1]'_{1/2})[2,1]_{J^\pi} \times [2,1]''_{1/2})[3]$

$$-\frac{1}{2}\left[(-|211\rangle_{L^\pi}|211\rangle'_{1/2} + |121\rangle_{L^\pi}|121\rangle'_{1/2})_{J^\pi}|211\rangle''_{1/2}\right.$$
$$\left.+ (|211\rangle_{L^\pi}|121\rangle'_{1/2} + |121\rangle_{L^\pi}|211\rangle'_{1/2})_{J^\pi}|121\rangle''_{1/2}\right]$$

(4) $(([1^3]_{L^\pi} \times [2,1]'_{1/2})[2,1]_{J^\pi} \times [2,1]''_{1/2})[3]$

$$\frac{1}{\sqrt{2}}\left[-(|321\rangle_{L^\pi}|121\rangle'_{1/2})_{J^\pi}|211\rangle''_{1/2} + (|321\rangle_{L^\pi}|211\rangle'_{1/2})_{J^\pi}|121\rangle''_{1/2}\right]$$

(5) $(([3]_{L^\pi} \times [3]'_{3/2})[3]_{J^\pi} \times [3]''_{3/2})[3]$

$$\left[(|111\rangle_{L^\pi}|111\rangle'_{3/2})_{J^\pi}|111\rangle''_{3/2}\right]$$

(6) $(([2,1]_{L^\pi} \times [2,1]'_{1/2})[3]_{J^\pi} \times [3]''_{3/2})[3]$

$$\frac{1}{\sqrt{2}}\left[(|211\rangle_{L^\pi}|211\rangle'_{1/2} + |121\rangle_{L^\pi}|121\rangle'_{1/2})_{J^\pi}|111\rangle''_{3/2}\right]$$

Table 3.4 The Jacobi coordinates, their conjugate momenta, and the inverse transformations for three, equal mass particles

$$\underset{\sim}{R} = 1/3(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3)$$

$$\underset{\sim}{\lambda} = 1/\sqrt{6}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 - 2\underset{\sim}{r}_3)$$

$$\underset{\sim}{\rho} = 1/\sqrt{2}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)$$

$$\underset{\sim}{P}_R = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 + \underset{\sim}{p}_3$$

$$\underset{\sim}{P}_\lambda = 1/\sqrt{6}(\underset{\sim}{p}_1 + \underset{\sim}{p}_2 - 2\underset{\sim}{p}_3)$$

$$\underset{\sim}{P}_\rho = 1/\sqrt{2}(\underset{\sim}{p}_1 - \underset{\sim}{p}_2)$$

$$\underset{\sim}{r}_1 = \underset{\sim}{R} + 1/\sqrt{6}\ \underset{\sim}{\lambda} + 1/\sqrt{2}\ \underset{\sim}{\rho}$$

$$\underset{\sim}{r}_2 = \underset{\sim}{R} + 1/\sqrt{6}\ \underset{\sim}{\lambda} - 1/\sqrt{2}\ \underset{\sim}{\rho}$$

$$\underset{\sim}{r}_3 = \underset{\sim}{R} - 2/\sqrt{6}\ \underset{\sim}{\lambda}$$

$$\underset{\sim}{p}_1 = 1/3\ \underset{\sim}{P}_R + 1/\sqrt{6}\ \underset{\sim}{P}_\lambda + 1/\sqrt{2}\ \underset{\sim}{P}_\rho$$

$$\underset{\sim}{p}_2 = 1/3\ \underset{\sim}{P}_R + 1/\sqrt{6}\ \underset{\sim}{P}_\lambda - 1/\sqrt{2}\ \underset{\sim}{P}_\rho$$

$$\underset{\sim}{p}_3 = 1/3\ \underset{\sim}{P}_R - 2/\sqrt{6}\ \underset{\sim}{P}_\lambda$$

of mass and $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ are the internal coordinates. Note that under interchange of indices, $\underset{\sim}{R}$ transforms as the $|111\rangle$ basis function of the [3] irrep and $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ transform as the $|211\rangle$ and $|121\rangle$ basis functions of the [2,1] irrep, respectively. The center of mass frame is chosen, $\underset{\sim}{P}_R = 0$, and the baryon is centered at the origin so that on the average, $\underset{\sim}{R} = 0$. Hence,

$$\underset{\sim}{L} = \underset{\sim}{\ell}_1 + \underset{\sim}{\ell}_2 + \underset{\sim}{\ell}_3 = \underset{\sim}{L}_\lambda + \underset{\sim}{L}_\rho \quad .$$

Thus, when H is written in terms of the Jacobi variables, the terms involving only $\underset{\sim}{\lambda}$, $\underset{\sim}{P}_\lambda$, $\underset{\sim}{\rho}$, and $\underset{\sim}{P}_\rho$ will be nonzero in (3.2). This implies that the space functions need to be given in terms of only $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$.

These Jacobi coordinates have the property that

$$\frac{1}{2m}\,(p_1^2 + p_2^2 + p_3^2)$$

is transformed to

$$\frac{P_R^2}{2M_R} + \frac{1}{2m}\,(P_\lambda^2 + P_\rho^2) \quad ,$$

where $M_R = 3m$. This shows that the mass of the u and d quarks, m, may be associated with the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ variables. The complete set of space functions is then chosen to be the two-particle, harmonic oscillator functions in the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ variables, where each particle has the same mass, m. These are the set of functions, coupled to total orbital angular momentum L,

$$\{|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> = \sum_{M_\lambda M_\rho} C(L_\lambda L_\rho L;\ M_\lambda M_\rho M)\,|N_\lambda L_\lambda M_\lambda>|N_\rho L_\rho M_\rho>:$$

$$N_\lambda,L_\lambda,N_\rho,L_\rho = 0,1,2,\ \ldots,\ L \geq |L_\lambda - L_\rho|,\ L \leq L_\lambda + L_\rho,$$

$$\text{and } M_L = -L,-L+1,\ \ldots,\ L\} \quad , \tag{3.8}$$

where the single particle, harmonic oscillator functions are given by (9.3) and (9.4). The inverse length parameter, β, is the same for the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ harmonic oscillator functions and

$$\beta = 1/b = \sqrt{m\omega} \quad , \tag{3.9}$$

where b is the length parameter of (9.3). By (9.1), these functions are eigenfunctions of

$$H = \frac{1}{2m}(P_\lambda^2 + P_\rho^2) + \frac{1}{2}m\omega^2(\lambda^2 + \rho^2) \quad , \tag{3.10}$$

where ω is the oscillator frequency, and

$$H|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> = \omega(2N_\lambda + L_\lambda + 2N_\rho + L_\rho + 3)|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> \ . \tag{3.11}$$

Thus, the excitation energy of $|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L>$, in units of ω, is $2N_\lambda + L_\lambda + 2N_\rho + L_\rho$. Under reflections,

$$P_\pi \begin{bmatrix} \underset{\sim}{\lambda} \\ \underset{\sim}{\rho} \end{bmatrix} = \begin{bmatrix} -\underset{\sim}{\lambda} \\ -\underset{\sim}{\rho} \end{bmatrix}$$

and since

$$Y_{LM}(-\hat{r}) = (-1)^L Y_{LM}(\hat{r}) \quad ,$$

then

$$P_\pi |(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> = (-1)^{L_\lambda + L_\rho} |(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> \quad . \qquad (3.12)$$

Hence, the negative parity states have an odd number of units of excitation energy and the positive parity have an even number. The basis is truncated by using all states with an excitation energy less than or equal to $n\omega$, where n is some finite integer. In this thesis, a 6ω basis is used, but the 2ω and 4ω bases will be used also in order that the convergence of eigenvalues may be investigated as the size of the basis increases. Table 3.5 lists the 6ω basis. The notation is

$$[K,\pi> = [(N_\lambda L_\lambda, N_\rho L_\rho)L:E> \quad ,$$

where K is the index and two sets of indices are used, one for $\pi = -1$ and one for $\pi = +1$. The excitation energy, in units of ω, is given by $E = 2N_\lambda + L_\lambda + 2N_\rho + L_\rho$.

The next step is to extract the S_3 irreps, under interchange of indices 1, 2, and 3, from this set. The method given in Appendix B, utilizing the S_3 projection operators and the Gram-Schmit orthonormalization procedure, will be used. Before doing this, the S_3 projection operators of Table 8.3 will be rewritten in a form which is easier to utilize. By Table 8.2, where $\underset{\sim}{\lambda} = |211>$ and $\underset{\sim}{\rho} = |121>$,

$$(e) \begin{bmatrix} \underset{\sim}{\lambda} \\ \underset{\sim}{\rho} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\lambda} \\ \underset{\sim}{\rho} \end{bmatrix}$$

and

Table 3.5. The 6ω basis of two-particle, harmonic oscillator states, $[K,\pi\rangle = [(N_\lambda L_\lambda, N_\rho L_\rho)L:E\rangle$

$[1,+\rangle = [(0\ 0,0\ 0)0:0\rangle$
$[2,+\rangle = [(1\ 0,0\ 0)0:2\rangle$
$[3,+\rangle = [(0\ 0,1\ 0)0:2\rangle$
$[4,+\rangle = [(0\ 1,0\ 1)0:2\rangle$
$[5,+\rangle = [(0\ 1,0\ 1)1:2\rangle$
$[6,+\rangle = [(0\ 2,0\ 0)2:2\rangle$
$[7,+\rangle = [(0\ 0,0\ 2)2:2\rangle$
$[8,+\rangle = [(0\ 1,0\ 1)2:2\rangle$
$[9,+\rangle = [(0\ 0,2\ 0)0:4\rangle$
$[10,+\rangle = [(2\ 0,0\ 0)0:4\rangle$
$[11,+\rangle = [(0\ 1,1\ 1)0:4\rangle$
$[12,+\rangle = [(1\ 1,0\ 1)0:4\rangle$
$[13,+\rangle = [(1\ 0,1\ 0)0:4\rangle$
$[14,+\rangle = [(0\ 2,0\ 2)0:4\rangle$
$[15,+\rangle = [(0\ 1,1\ 1)1:4\rangle$
$[16,+\rangle = [(1\ 1,0\ 1)1:4\rangle$
$[17,+\rangle = [(0\ 2,0\ 2)1:4\rangle$
$[18,+\rangle = [(0\ 0,1\ 2)2:4\rangle$
$[19,+\rangle = [(1\ 2,0\ 0)2:4\rangle$
$[20,+\rangle = [(0\ 1,1\ 1)2:4\rangle$
$[21,+\rangle = [(1\ 1,0\ 1)2:4\rangle$
$[22,+\rangle = [(0\ 1,0\ 3)2:4\rangle$
$[23,+\rangle = [(0\ 3,0\ 1)2:4\rangle$
$[24,+\rangle = [(1\ 0,0\ 2)2:4\rangle$
$[25,+\rangle = [(0\ 2,1\ 0)2:4\rangle$
$[26,+\rangle = [(0\ 2,0\ 2)2:4\rangle$
$[27,+\rangle = [(0\ 1,0\ 3)3:4\rangle$
$[28,+\rangle = [(0\ 3,0\ 1)3:4\rangle$
$[29,+\rangle = [(0\ 2,0\ 2)3:4\rangle$
$[30,+\rangle = [(0\ 0,0\ 4)4:4\rangle$
$[31,+\rangle = [(0\ 4,0\ 0)4:4\rangle$
$[32,+\rangle = [(0\ 1,0\ 3)4:4\rangle$
$[33,+\rangle = [(0\ 3,0\ 1)4:4\rangle$
$[34,+\rangle = [(0\ 2,0\ 2)4:4\rangle$
$[35,+\rangle = [(0\ 0,3\ 0)0:6\rangle$
$[36,+\rangle = [(3\ 0,0\ 0)0:6\rangle$
$[37,+\rangle = [(0\ 1,2\ 1)0:6\rangle$
$[38,+\rangle = [(2\ 1,0\ 1)0:6\rangle$
$[39,+\rangle = [(1\ 0,2\ 0)0:6\rangle$
$[40,+\rangle = [(2\ 0,1\ 0)0:6\rangle$
$[41,+\rangle = [(0\ 2,1\ 2)0:6\rangle$
$[42,+\rangle = [(1\ 2,0\ 2)0:6\rangle$
$[43,+\rangle = [(1\ 1,1\ 1)0:6\rangle$
$[44,+\rangle = [(0\ 3,0\ 3)0:6\rangle$
$[45,+\rangle = [(0\ 1,2\ 1)1:6\rangle$
$[46,+\rangle = [(2\ 1,0\ 1)1:6\rangle$
$[47,+\rangle = [(0\ 2,1\ 2)1:6\rangle$
$[48,+\rangle = [(1\ 2,0\ 2)1:6\rangle$
$[49,+\rangle = [(1\ 1,1\ 1)1:6\rangle$
$[50,+\rangle = [(0\ 3,0\ 3)1:6\rangle$
$[51,+\rangle = [(0\ 0,2\ 2)2:6\rangle$
$[52,+\rangle = [(2\ 2,0\ 0)2:6\rangle$
$[53,+\rangle = [(0\ 1,2\ 1)2:6\rangle$
$[54,+\rangle = [(2\ 1,0\ 1)2:6\rangle$
$[55,+\rangle = [(0\ 1,1\ 3)2:6\rangle$
$[56,+\rangle = [(1\ 3,0\ 1)2:6\rangle$
$[57,+\rangle = [(1\ 0,1\ 2)2:6\rangle$
$[58,+\rangle = [(1\ 2,1\ 0)2:6\rangle$
$[59,+\rangle = [(0\ 2,2\ 0)2:6\rangle$
$[60,+\rangle = [(2\ 0,0\ 2)2:6\rangle$
$[61,+\rangle = [(0\ 2,1\ 2)2:6\rangle$
$[62,+\rangle = [(1\ 2,0\ 2)2:6\rangle$
$[63,+\rangle = [(0\ 2,0\ 4)2:6\rangle$
$[64,+\rangle = [(0\ 4,0\ 2)2:6\rangle$
$[65,+\rangle = [(1\ 1,1\ 1)2:6\rangle$
$[66,+\rangle = [(1\ 1,0\ 3)2:6\rangle$
$[67,+\rangle = [(0\ 3,1\ 1)2:6\rangle$
$[68,+\rangle = [(0\ 3,0\ 3)2:6\rangle$
$[69,+\rangle = [(0\ 1,1\ 3)3:6\rangle$
$[70,+\rangle = [(1\ 3,0\ 1)3:6\rangle$
$[71,+\rangle = [(0\ 2,1\ 2)3:6\rangle$
$[72,+\rangle = [(1\ 2,0\ 2)3:6\rangle$
$[73,+\rangle = [(0\ 2,0\ 4)3:6\rangle$
$[74,+\rangle = [(0\ 4,0\ 2)3:6\rangle$
$[75,+\rangle = [(1\ 1,0\ 3)3:6\rangle$
$[76,+\rangle = [(0\ 3,1\ 1)3:6\rangle$
$[77,+\rangle = [(0\ 3,0\ 3)3:6\rangle$
$[78,+\rangle = [(0\ 0,1\ 4)4:6\rangle$
$[79,+\rangle = [(1\ 4,0\ 0)4:6\rangle$
$[80,+\rangle = [(0\ 1,1\ 3)4:6\rangle$
$[81,+\rangle = [(1\ 3,0\ 1)4:6\rangle$
$[82,+\rangle = [(0\ 1,0\ 5)4:6\rangle$
$[83,+\rangle = [(0\ 5,0\ 1)4:6\rangle$
$[84,+\rangle = [(1\ 0,0\ 4)4:6\rangle$
$[85,+\rangle = [(0\ 4,1\ 0)4:6\rangle$
$[86,+\rangle = [(0\ 2,1\ 2)4:6\rangle$
$[87,+\rangle = [(1\ 2,0\ 2)4:6\rangle$
$[88,+\rangle = [(0\ 2,0\ 4)4:6\rangle$
$[89,+\rangle = [(0\ 4,0\ 2)4:6\rangle$
$[90,+\rangle = [(1\ 1,0\ 3)4:6\rangle$
$[91,+\rangle = [(0\ 3,1\ 1)4:6\rangle$
$[92,+\rangle = [(0\ 3,0\ 3)4:6\rangle$
$[93,+\rangle = [(0\ 1,0\ 5)5:6\rangle$
$[94,+\rangle = [(0\ 5,0\ 1)5:6\rangle$
$[95,+\rangle = [(0\ 2,0\ 4)5:6\rangle$
$[96,+\rangle = [(0\ 4,0\ 2)5:6\rangle$
$[97,+\rangle = [(0\ 3,0\ 3)5:6\rangle$
$[98,+\rangle = [(0\ 0,0\ 6)6:6\rangle$
$[99,+\rangle = [(0\ 6,0\ 0)6:6\rangle$
$[100,+\rangle = [(0\ 1,0\ 5)6:6\rangle$
$[101,+\rangle = [(0\ 5,0\ 1)6:6\rangle$
$[102,+\rangle = [(0\ 2,0\ 4)6:6\rangle$
$[103,+\rangle = [(0\ 4,0\ 2)6:6\rangle$
$[104,+\rangle = [(0\ 3,0\ 3)6:6\rangle$
$[1,-\rangle = [(0\ 1,0\ 0)1:1\rangle$
$[2,-\rangle = [(0\ 0,0\ 1)1:1\rangle$
$[3,-\rangle = [(0\ 0,1\ 1)1:3\rangle$
$[4,-\rangle = [(1\ 1,0\ 0)1:3\rangle$
$[5,-\rangle = [(1\ 0,0\ 1)1:3\rangle$
$[6,-\rangle = [(0\ 1,1\ 0)1:3\rangle$
$[7,-\rangle = [(0\ 2,0\ 1)1:3\rangle$
$[8,-\rangle = [(0\ 1,0\ 2)1:3\rangle$
$[9,-\rangle = [(0\ 2,0\ 1)2:3\rangle$
$[10,-\rangle = [(0\ 1,0\ 2)2:3\rangle$
$[11,-\rangle = [(0\ 0,0\ 3)3:3\rangle$
$[12,-\rangle = [(0\ 3,0\ 0)3:3\rangle$
$[13,-\rangle = [(0\ 2,0\ 1)3:3\rangle$
$[14,-\rangle = [(0\ 1,0\ 2)3:3\rangle$
$[15,-\rangle = [(0\ 0,2\ 1)1:5\rangle$
$[16,-\rangle = [(2\ 1,0\ 0)1:5\rangle$
$[17,-\rangle = [(0\ 1,2\ 0)1:5\rangle$
$[18,-\rangle = [(2\ 0,0\ 1)1:5\rangle$
$[19,-\rangle = [(0\ 1,1\ 2)1:5\rangle$
$[20,-\rangle = [(1\ 2,0\ 1)1:5\rangle$
$[21,-\rangle = [(1\ 0,1\ 1)1:5\rangle$
$[22,-\rangle = [(1\ 1,1\ 0)1:5\rangle$
$[23,-\rangle = [(0\ 2,1\ 1)1:5\rangle$
$[24,-\rangle = [(1\ 1,0\ 2)1:5\rangle$
$[25,-\rangle = [(0\ 2,0\ 3)1:5\rangle$
$[26,-\rangle = [(0\ 3,0\ 2)1:5\rangle$
$[27,-\rangle = [(0\ 1,1\ 2)2:5\rangle$
$[28,-\rangle = [(1\ 2,0\ 1)2:5\rangle$
$[29,-\rangle = [(0\ 2,1\ 1)2:5\rangle$
$[30,-\rangle = [(1\ 1,0\ 2)2:5\rangle$
$[31,-\rangle = [(0\ 2,0\ 3)2:5\rangle$
$[32,-\rangle = [(0\ 3,0\ 2)2:5\rangle$
$[33,-\rangle = [(0\ 0,1\ 3)3:5\rangle$
$[34,-\rangle = [(1\ 3,0\ 0)3:5\rangle$
$[35,-\rangle = [(0\ 1,1\ 2)3:5\rangle$
$[36,-\rangle = [(1\ 2,0\ 1)3:5\rangle$
$[37,-\rangle = [(0\ 1,0\ 4)3:5\rangle$
$[38,-\rangle = [(0\ 4,0\ 1)3:5\rangle$
$[39,-\rangle = [(1\ 0,0\ 3)3:5\rangle$
$[40,-\rangle = [(0\ 3,1\ 0)3:5\rangle$
$[41,-\rangle = [(0\ 2,1\ 1)3:5\rangle$
$[42,-\rangle = [(1\ 1,0\ 2)3:5\rangle$
$[43,-\rangle = [(0\ 2,0\ 3)3:5\rangle$
$[44,-\rangle = [(0\ 3,0\ 2)3:5\rangle$
$[45,-\rangle = [(0\ 1,0\ 4)4:5\rangle$
$[46,-\rangle = [(0\ 4,0\ 1)4:5\rangle$
$[47,-\rangle = [(0\ 2,0\ 3)4:5\rangle$
$[48,-\rangle = [(0\ 3,0\ 2)4:5\rangle$
$[49,-\rangle = [(0\ 0,0\ 5)5:5\rangle$
$[50,-\rangle = [(0\ 5,0\ 0)5:5\rangle$
$[51,-\rangle = [(0\ 1,0\ 4)5:5\rangle$
$[52,-\rangle = [(0\ 4,0\ 1)5:5\rangle$
$[53,-\rangle = [(0\ 2,0\ 3)5:5\rangle$
$[54,-\rangle = [(0\ 3,0\ 2)5:5\rangle$

$$(12)\begin{bmatrix}\lambda\\ \sim\\ \rho\\ \sim\end{bmatrix} = \begin{bmatrix}\lambda\\ \sim\\ -\rho\\ \sim\end{bmatrix} \quad .$$

Thus, because $Y_{L_\lambda M_\rho}(-\hat{\rho}) = (-1)^{L_\rho} Y_{L_\rho M_\rho}(\hat{\rho})$,

$$1/2[(e) + (12)]|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> = |(N_\lambda L_\lambda N_\rho L_\rho)L,M_L>, \text{ if } L_\rho \text{ is even}$$

$$= 0 \quad , \quad \text{if } L_\rho \text{ is odd} \qquad (3.13)$$

and

$$1/2[(e) - (12)]|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L> = |(N_\lambda L_\lambda N_\rho L_\rho)L,M_L>, \text{ if } L_\rho \text{ is odd}$$

$$= 0 \quad , \quad \text{if } L_\rho \text{ is even} \quad . \qquad (3.14)$$

The operator $1/2[(e) + (12)]$ projects out the functions of Table 3.5 with even L_ρ and it will be denoted as P^E_ρ. $1/2[(e) - (12)]$ projects out those with odd L_ρ and it will be denoted as P^O_ρ. Note that under the subgroup S_2, operating on indices 1 and 2 only, P^E_ρ and P^O_ρ are $P^{[2]}_{11,11}$ and $P^{[1^2]}_{21,21}$ of Table 8.2, respectively. Then, because all elements of S_3 may be written as products of (e), (12), and (23) only, the projection operators of Table 8.3 may be written in terms of P^E_ρ, P^O_ρ, and (23). These are given in Table 3.6. The result of operating on the states with the (23) operator is needed next. By Table 8.2,

$$(23)\begin{bmatrix}\lambda\\ \sim\\ \rho\\ \sim\end{bmatrix} = \begin{bmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}\begin{bmatrix}\lambda\\ \sim\\ \rho\\ \sim\end{bmatrix} \quad , \qquad (3.15)$$

Table 3.6. The S_3 projection operators of Table 8.3 written in terms of P^E_ρ, P^0_ρ, and (23)

$$P^{[3]}_{111,111} = 1/3\{P^E_\rho + 2P^E_\rho(23)P^E_\rho\}$$

$$P^{[2,1]}_{211,211} = 2/3\{P^E_\rho - P^E_\rho(23)P^E_\rho\}$$

$$P^{[2,1]}_{121,211} = 2/\sqrt{3}\{P^0_\rho(23)P^E_\rho\}$$

$$P^{[1^3]}_{321,321} = 1/3\{P^0_\rho - 2P^0_\rho(23)P^0_\rho\}$$

and then, since the states of (3.8) are a complete set, one may write

$$(23)\,|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L\rangle = \sum_{\substack{n_\lambda \ell_\lambda n_\rho \ell_\rho \\ \ell m_\ell}} \langle N_\lambda L_\lambda N_\rho L_\rho L M_L | n_\lambda \ell_\lambda n_\rho \ell_\rho \ell m_\ell\rangle \times |(n_\lambda \ell_\lambda n_\rho \ell_\rho)\ell, m_\ell\rangle \quad , \tag{3.16}$$

where $|(n_\lambda \ell_\lambda n_\rho \ell_\rho)\ell,m_\ell\rangle$ are again the two-particle, harmonic oscillator states of (3.8), depending on the variables $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$. The coefficients of the unitary transformation, $\langle N_\lambda L_\lambda N_\rho L_\rho L M_L | n_\lambda \ell_\lambda n_\rho \ell_\rho \ell m_\ell\rangle$, are Talmi brackets. The explicit form of these is derived in Appendix C and given by (9.22), where $-a = d = 1/2$ and $b = c = \sqrt{3}/2$. By (9.16)-(9.19), the Talmi brackets are zero unless

$$L = \ell \quad ,$$

$$M_L = m_\ell \quad ,$$

$$2N_\lambda + L_\lambda + 2N_\rho + L_\rho = 2n_\lambda + \ell_\lambda + 2n_\rho + \ell_\rho \quad ,$$

and

$$(-1)^{L_\lambda + L_\rho} = (-1)^{\ell_\lambda + \ell_\rho} \quad .$$

Thus, $(23)\,|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L\rangle$ is a linear combination of $|(n_\lambda \ell_\lambda n_\rho \ell_\rho)\ell,m_\ell\rangle$, all having the same excitation energy and, hence, the same parity, as $|(N_\lambda L_\lambda N_\rho L_\rho)L,M_L\rangle$.

Then, the space S_3 irreps are generated from the states of Table 3.5 by using the following steps.

(1) For each L and E values, take the set from Table 3.5,

$$F = \{[K,\pi\rangle = [(N_\lambda L_\lambda, N_\rho L_\rho)L:E\rangle: \text{ all } N_\lambda, L_\lambda, N_\rho, \text{ and } L_\rho \text{ for fixed L and E}\} \quad ,$$

and apply each projection operator of Table 3.6 to each state in F. This gives the set

$$G = \{P^j_{k,\ell}[K,\pi\rangle: \text{ all } [K,\pi\rangle \text{ in F and all } P^j_{k,\ell}\} \quad ,$$

each element being a linear combination of the states in F.

(2) As detailed in Appendix C, take each of the four sets

$$G(j;k) = \{P^j_{k,\ell}[K,\pi\rangle: \text{ all } [K,\pi\rangle \text{ in F where } P^j_{k,\ell}[K,\pi\rangle \neq 0\}$$

and apply the Gram-Schmit orthonormalization procedure to each of the four sets. The orthonormal set extracted from G(j;k) will be the $|k\rangle$ basis functions of the j irrep. The total number of $|k\rangle$ in the j irrep, extracted in this manner, equals the number of occurrences of the irrep j in the set F. Note that this amounts to applying a unitary transformation to F and, hence, the number of states in F must equal the resulting number of orthonormal, S_3 basis functions.

(3) Repeat for the next values of L and E until all the states in Table 3.5 have been exhausted.

As an example, we will generate the S_3 basis functions from the set of states in Table 3.5 with L = E = 4,

$$F = \begin{bmatrix} [30,+\rangle \\ [31,+\rangle \\ [32,+\rangle \\ [33,+\rangle \\ [34,+\rangle \end{bmatrix} \quad .$$

Using (3.16) and (9.22),

$$(23)F = \frac{1}{16}\begin{bmatrix} 1 & 9 & 2\sqrt{3} & 6\sqrt{3} & 3\sqrt{6} \\ 9 & 1 & -6\sqrt{3} & -2\sqrt{3} & 3\sqrt{6} \\ 2\sqrt{3} & -6\sqrt{3} & 8 & 0 & 6\sqrt{2} \\ 6\sqrt{3} & -2\sqrt{3} & 0 & 8 & -6\sqrt{2} \\ 3\sqrt{6} & 3\sqrt{6} & 6\sqrt{2} & -6\sqrt{2} & -2 \end{bmatrix} F \quad . \tag{3.17}$$

Then, by using (3.13), (3.14), and (3.17), it may be shown that

$$P^{[3]}_{111,111}F = \frac{1}{8}\begin{bmatrix} 3 & 3 & 0 & 0 & \sqrt{6} \\ 3 & 3 & & & \sqrt{6} \\ 0 & & & & \\ 0 & & & & \\ \sqrt{6} & \sqrt{6} & 0 & 0 & 2 \end{bmatrix} F \quad , \tag{3.18}$$

$$P^{[2,1]}_{211,211}F = \frac{1}{8}\begin{bmatrix} 5 & -3 & 0 & 0 & -\sqrt{6} \\ -3 & 5 & & & -\sqrt{6} \\ 0 & & & & \\ 0 & & & & \\ -\sqrt{6} & -\sqrt{6} & 0 & 0 & 6 \end{bmatrix} F \quad , \tag{3.19}$$

$$P^{[2,1]}_{121,211}F = \frac{1}{4}\begin{bmatrix} 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & -3 & -1 & 0 \\ 0 & & & & \\ 0 & & & & \\ 0 & 0 & \sqrt{6} & \sqrt{6} & 0 \end{bmatrix} F \quad , \tag{3.20}$$

and

$$P^{[1^3]}_{321,321}F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad , \tag{3.21}$$

where any element not given explicitly in the 5 x 5 matrices is assumed

to be zero. By (3.21), the $[1^3]$ irrep does not occur in F. Applying the Gram-Schmit procedure, operating first on row one, then row two, etc., gives,

$$\frac{1}{4}\begin{bmatrix} \sqrt{6} & \sqrt{6} & 0 & 0 & 2 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} F \qquad (3.22)$$

for (3.18),

$$\frac{1}{20}\begin{bmatrix} 5\sqrt{10} & -3\sqrt{10} & 0 & 0 & -2\sqrt{15} \\ 0 & 4\sqrt{10} & 0 & 0 & -4\sqrt{15} \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} F \qquad (3.23)$$

for (3.19), and

$$\frac{1}{10}\begin{bmatrix} 0 & 0 & \sqrt{10} & 3\sqrt{10} & 0 \\ 0 & 0 & -3\sqrt{10} & \sqrt{10} & 0 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} F \qquad (3.24)$$

for (3.20). From (3.22), there is one occurrence of the [3] irrep and its basis function is

$$|111,1\rangle = 1/4[\sqrt{6}[30,+\rangle + \sqrt{6}[31,+\rangle + 2[34,+\rangle] \quad .$$

From (3.23) and (3.24), there are two occurrences of the [2,1] irrep. In the first occurrence,

$$|211,1\rangle = 1/20[5\sqrt{10}[30,+\rangle - 3\sqrt{10}[31,+\rangle - 2\sqrt{15}[34,+\rangle]$$

$$|121,1\rangle = 1/10[\sqrt{10}[32,+\rangle + 3\sqrt{10}[33,+\rangle] \quad ,$$

and in the second,

$$|211,2\rangle = 1/20[4\sqrt{10}[31,+\rangle - 4\sqrt{15}[34,+\rangle]$$

$$|121,2\rangle = 1/10[-3\sqrt{10}[32,+\rangle + \sqrt{10}[33,+\rangle] \quad .$$

Note that instead of applying the Gram-Schmit procedure to (3.20), the $|121\rangle$ functions could have been found from (3.23) by using $P^{[2,1]}_{121,211}$ since

$$P^{[2,1]}_{121,211}|211,i\rangle = |121,i\rangle \quad . \tag{3.25}$$

In the computer program which performs the task of extracting the S_3 irreps, the method shown in the above example is used for computing the $|121\rangle$ functions, not (3.25). This is because (3.25) uses the numerically generated matrix of Talmi brackets in (3.17) two times in computing $|121\rangle$, first in finding the $|211,i\rangle$ functions and then in determining $P^{[2,1]}_{121,211}|211,i\rangle$. Since this matrix involves more computer round-off error than the Gram-Schmit procedure, the method shown in the example, which uses the matrix only once to compute the $|121\rangle$ functions, is used.

The lengthy table of the S_3 irreps, extracted from the states of Table 3.5, is given in Appendix D, Table 10.1. The notation used is

$$[m\alpha,\pi\rangle = [k,i:E,L\rangle = \sum_K C^{k,i}_K [K,\pi\rangle \quad ,$$

where m is the index and two sets of indices are used, one for $\pi = +1$ and one for $\pi = -1$. The label α differentiates the two basis functions of the [2,1] irrep; $\alpha = a$ for the $|211\rangle$ function and $\alpha = b$ for the

$|121\rangle$. The label k is the S_3 basis function label and i labels the irrep occurrence for one set of L and E values. The $[K,\pi\rangle$ are the states of Table 3.5 and the $C_K^{k,i}$ are the coefficients of the unitary transformation.

These S_3 irreps are then used to generate the different sets of basis functions. Each set will consist of all possible functions with the same τ values (1/2 for N, 3/2 for Δ) and the same J^π values such that the space-spin-isospin function is overall symmetric. The sets will then be labeled by N J^π or Δ J^π for the various J^π values. Thus, one set of basis functions consists of the two spin irreps of Table 3.2, the isospin irrep of Table 3.1 with isospin τ, and all the space irreps, with parity π, of Table 10.1, which may be coupled to a symmetric function using the rules of Table 3.3, and which may be coupled to total angular momentum J. As an example, we will give the basis of states for the N $1/2^+$ resonances for $E \leq 2$. If $S = 3/2$, then $L^\pi = 1^+$ and 2^+ and the space irrep must be [2,1]. The only function in Table 10.1 satisfying this is $[6,+\rangle$. If $S = 1/2$, then $L^\pi = 0^+$ and 1^+ and the space irreps may be [3], [2,1], or $[1^3]$. The functions satisfying this are $[1,+\rangle$, $[2,+\rangle$, $[3,+\rangle$, and $[4,+\rangle$. The five states are then, in the notation of Table 3.3,

$$|1\rangle = \frac{1}{\sqrt{2}}\left[\left([1,+\rangle_{0^+}|211\rangle'_{1/2}\right)_{1/2^+}|211\rangle''_{1/2} + \left([1,+\rangle_{0^+}|121\rangle'_{1/2}\right)_{1/2^+} \times |121\rangle''_{1/2}\right] \quad \{2\} \quad ,$$

$$|2\rangle = \frac{1}{\sqrt{2}}\left[\left([2,+\rangle_{0^+}|211\rangle'_{1/2}\right)_{1/2^+}|211\rangle''_{1/2} + \left([2,+\rangle_{0^+}|121\rangle'_{1/2}\right)_{1/2^+} \times |121\rangle''_{1/2}\right] \quad \{2\} \quad ,$$

$$|3\rangle = -\frac{1}{2}\left[\left(-[3a,+\rangle_{0^+}|211\rangle'_{1/2} + [3b,+\rangle_{0^+}|121\rangle'_{1/2}\right)_{1/2^+}|211\rangle''_{1/2} + \left([3a,+\rangle_{0^+}|121\rangle'_{1/2} + [3b,+\rangle_{0^+}|211\rangle'_{1/2}\right)_{1/2^+}|121\rangle''_{1/2}\right] \quad \{3\} \quad ,$$

$$|4\rangle = \frac{1}{\sqrt{2}}\left[-\left([4,+\rangle_{1^+}|121\rangle'_{1/2}\right)_{1/2^+}|211\rangle''_{1/2} + \left([4,+\rangle_{1^+}|211\rangle'_{1/2}\right)_{1/2^+} \times |121\rangle''_{1/2}\right] \quad \{4\} \quad ,$$

and

$$|5\rangle = \frac{1}{\sqrt{2}}\left[\left([6a,+\rangle_{2^+}|111\rangle'_{3/2}\right)_{1/2^+}|211\rangle''_{1/2} + \left([6b,+\rangle_{2^+}|111\rangle'_{3/2}\right)_{1/2^+} \times |121\rangle''_{1/2}\right] \quad \{1\} \quad ,$$

where the number in brackets refers to the coupling rule in Table 3.3 which was used. Diagonalizing H in this basis then gives an approximation for the five lowest energy eigenvalues and corresponding eigenvectors of the N $1/2^+$ resonances of H. Table 10.2 lists all the 6ω bases for the N and Δ, where J may be 1/2, 3/2, ..., 15/2 for $\pi = +1$ and J may be 1/2, 3/2, ..., 13/2 for $\pi = -1$. The notation used in Table 10.2 is $[l\rangle = [([m,\pi\rangle,[S\rangle)J\rangle[\tau\rangle$, where l is the index for one set of J, π, and τ and $[m,\pi\rangle$ is the space irrep of Table 10.1. The label S is the total spin, where $[1/2\rangle$ is the [2,1] irrep and $[3/2\rangle$ is the [3] irrep. Similarly, $[\tau\rangle = [1/2\rangle$ and $[\tau\rangle = [3/2\rangle$ are the [2,1] and [3] irreps, respectively. In this notation, the five previous N $1/2^+$ states are

$$|1\rangle = [([1,+\rangle,[1/2\rangle)1/2\rangle[1/2\rangle \quad ,$$

$$|2\rangle = [([2,+\rangle,[1/2\rangle)1/2\rangle[1/2\rangle \quad ,$$

$$|3\rangle = [([3,+\rangle,[1/2\rangle)1/2\rangle[1/2\rangle \quad ,$$

$$|4\rangle = [([4,+\rangle,[1/2\rangle)1/2\rangle[1/2\rangle \quad ,$$

and

$$|5\rangle = [([6,+\rangle,[3/2\rangle)1/2\rangle[1/2\rangle \quad .$$

This is a shorthand notation, but the coupling rule of Table 3.3 which has been used to give the overall symmetric function may be inferred once the irrep label of $[m,\pi\rangle$ is found from Table 10.1.

IV. THE PROCESSING OF THE HAMILTONIAN

In this chapter, the diagonalization of H will be discussed. This diagonalization is done in a truncated basis and yields approximate eigenvectors and eigenvalues of H. Also, magnetic moments and charge radii will be calculated for the lowest energy N $1/2^+$ and Δ $3/2^+$ states. To accomplish this, formulas for the matrix elements of H, the magnetic moment operator and the charge radius operator, are needed.

By using Tables 3.1-3.3, 3.5 and 10.1-10.2, we may write a basis state for the N or Δ J^π resonances as

$$|\psi_n\rangle = \sum_{N_\lambda L_\lambda N_\rho L_\rho S_{12} \tau_{12}} B(n;\, N_\lambda L_\lambda N_\rho L_\rho S_{12} \tau_{12}) \,|[(N_\lambda L_\lambda N_\rho L_\rho) L (S_{12}) S] JM\rangle \times |(\tau_{12}) \tau M_\tau\rangle \quad . \tag{4.1}$$

Here, $|[(N_\lambda L_\lambda N_\rho L_\rho) L (S_{12}) S] JM\rangle$ denotes the angular momentum coupled state $|[|(N_\lambda L_\lambda N_\rho L_\rho) L\rangle |(S_{12}) S\rangle] JM\rangle$ and $B(n;\, N_\lambda L_\lambda N_\rho L_\rho S_{12} \tau_{12})$ is a coefficient in the sum over the space-spin-isospin states. In the sum, $2N_\lambda + L_\lambda + 2N_\rho + L_\rho$ is fixed, $L_\lambda + L_\rho$ is even if $\pi = +1$, and $L_\lambda + L_\rho$ is odd if $\pi = -1$. As an example, we will write state [2> in the N $1/2^+$ basis in this manner. Table 10.2 gives the form of this state as

$$[2\rangle = [([2,+\rangle, [1/2\rangle) 1/2\rangle [1/2\rangle \quad ,$$

and Tables 10.1 and 3.5 give the state [2,+> in the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ variables as

$$[2,+\rangle = 1/\sqrt{2}\, [|(10,00)0\rangle + |(00,10)0\rangle] \quad ,$$

where [2,+> is a [3] irrep. Using the second coupling rule of Table 3.3 and the form of the spin and isospin functions from Tables 3.1-3.2, we

find

$$[2> = 1/2[|[(10,00)0(1)1/2]1/2,M>|(1)1/2,M_\tau> + |[(00,10)0(1)1/2] \times 1/2,M>|(1)1/2,M_\tau> + |[(10,00)0(0)1/2]1/2,M>|(0)1/2,M_\tau> + |[(00,10)0(0)1/2]1/2,M>|(0)1/2,M_\tau>] \quad ,$$

which is the form desired. By using (4.1), we may write the matrix element of an operator, $\hat{0}$, between two basis states, $|\psi_m>$ and $|\psi_n>$, as

$$<\psi_m|\hat{0}|\psi_n> = \sum B(m;\ N'_\lambda L'_\lambda N'_\rho L'_\rho S'_{12} \tau'_{12}) \sum B(n;\ N_\lambda L_\lambda N_\rho L_\rho S_{12} \tau_{12}) \times <(\tau'_{12})\tau,M_\tau|<[(N'_\lambda L'_\lambda N'_\rho L'_\rho)L'(S'_{12})S']JM| \times \hat{0}|[(N_\lambda L_\lambda N_\rho L_\rho)L(S_{12})S]JM>|(\tau_{12})\tau,M_\tau> \quad . \tag{4.2}$$

Thus, only the fundamental matrix elements on the right hand side of (4.2) need to be calculated in order to find $<\psi_m|\hat{0}|\psi_n>$.

A. Matrix Elements of the Hamiltonian

By exploiting the property that the basis functions are symmetric in the combined space, spin, and isospin variables, the matrix elements of H may be greatly simplified. If

$$T_i = \frac{p_i^2}{2m} - \frac{p_i^4}{8m^3} \quad ,$$

then

$$<\psi_m|\sum_i T_i|\psi_n> = <\psi_m|(13)[(13)T_1(13)](13)|\psi_n> + <\psi_m|(23)[(23)T_2(23)] \times (23)|\psi_n> + <\psi_m|T_3|\psi_n> = 3<\psi_m|T_3|\psi_n> \quad . \tag{4.3}$$

Also, if

$$V_{ij} = V_{ij}^{LC} + V_{ij}^{OGE} \quad ,$$

then similarly

$$\langle\psi_m| \sum_{i>j} V_{ij} |\psi_n\rangle = 3\langle\psi_m|V_{12}|\psi_n\rangle \quad . \tag{4.4}$$

Thus, only matrix elements of T_3 and V_{12} need to be calculated. Physically, this means that since the u and d are identical particles in this model, all quarks must have equal kinetic energy matrix elements in an N or Δ resonance and the potential energy matrix elements between any two quarks must be equal. Results similar to (4.3) will be shown for other operators. Then, in terms of the Jacobi coordinates of Table 3.4, where we have set $\underset{\sim}{R} = \underset{\sim}{P}_R = 0$,

$$3T_3 = \frac{1}{m} P_\lambda^2 - \frac{1}{6m^3} P_\lambda^4 \quad , \tag{4.5}$$

$$3V_{12}^{LC} = 2\sqrt{2}\, k\rho \quad , \tag{4.6}$$

and

$$3V_{12}^{OGE} = -\sqrt{2}\, \alpha_S \rho^{-1} + \text{Darwin} + \text{Fermi-contact} + \text{spin-orbit} + \text{tensor-force} \, . \tag{4.7}$$

We have defined the terms in (4.7) as

$$\text{Darwin} = \frac{\alpha_S}{2\sqrt{2}\, m^2 \rho} \left[-\left(P_\rho^2 + \hat{\rho} \cdot (\hat{\rho} \cdot \underset{\sim}{P}_\rho)\underset{\sim}{P}_\rho\right) + \frac{1}{3}\left(P_\lambda^2 + (\hat{\rho} \cdot \underset{\sim}{P}_\lambda)^2\right)\right] \quad , \tag{4.8}$$

$$\text{Fermi-contact} = \frac{\pi\alpha_s}{\sqrt{2}\, m^2}\,\delta^3(\underset{\sim}{\rho})\left(1 + \frac{8}{3}\,\underset{\sim}{s}_1 \cdot \underset{\sim}{s}_2\right) \quad , \tag{4.9}$$

$$\text{spin-orbit} = \frac{\alpha_s}{2\sqrt{2}\, m^2}\left[3\rho^{-3}\underset{\sim}{L}_\rho \cdot \underset{\sim}{S}_{12} + \frac{1}{\sqrt{3}}\,\rho^{-2}(\hat{\rho} \times \underset{\sim}{P}_\lambda) \cdot (\underset{\sim}{s}_2 - \underset{\sim}{s}_1)\right] \quad , \tag{4.10}$$

and

$$\text{tensor-force} = \frac{\alpha_s}{\sqrt{2}\, m^2\rho^3}\left[3(\underset{\sim}{s}_1 \cdot \hat{\rho})(\underset{\sim}{s}_2 \cdot \hat{\rho}) - \underset{\sim}{s}_1 \cdot \underset{\sim}{s}_2\right] \quad , \tag{4.11}$$

in analogy with (2.6)-(2.9). Using (3.3) and the fact that H contains no isospin operators, we may show that

$$\begin{aligned}\langle(\tau'_{12})\tau, M_\tau|\langle[(N'_\lambda L'_\lambda N'_\rho L'_\rho)L'(S'_{12})S']JM|H|[(N_\lambda L_\lambda N_\rho L_\rho)L(S_{12})S]JM\rangle|(\tau_{12})\tau, M_\tau\rangle \\ = \delta_{\tau'_{12}\tau_{12}}\langle[(N'_\lambda L'_\lambda N'_\rho L'_\rho)L'(S'_{12})S']J||H||[(N_\lambda L_\lambda N_\rho L_\rho)L(S_{12})S]J\rangle \quad ,\end{aligned} \tag{4.12}$$

which further simplifies the matrix elements of H. The reduced matrix element in (4.12) will also be denoted in a shorthand notation as

$$\langle[L'S']J||H||[LS]J\rangle$$

in this chapter.

Next, we define a very useful function and determine its general formula. This function is defined as the integral over r^2dr of the radial parts of two harmonic oscillator functions multiplied by the mth power of r. This is the type of integral left in the matrix elements after integrating over the spin, isospin, and angular variables. By (9.3), this integral is

$$R(N'L',NL;\ m) \ = \ N_{N'L'}N_{NL} \int_0^\infty dr\ r^{L+L'+m+2} L_{N'}^{L'+1/2}(r^2) L_N^{L\ +1/2}(r^2) e^{-r^2} \quad ,$$

where β has been set equal to one and

$$N_{NL} \ = \ [2N!/\Gamma(N + L + 3/2)]^{1/2} \quad .$$

Substituting (9.4) for the Laguerre polynomials and integrating, we find

$$R(N'L',NL;\ m) \ = \ \left[\frac{\Gamma(N' + L' + 3/2)\Gamma(N+L+3/2)}{N'!N!}\right]^{1/2} \sum_{K'=0}^{N'} \sum_{K=0}^{N} (-1)^{K'+K}$$

$$\times \binom{N'}{K'} \binom{N}{K} \frac{\Gamma(P + K' + K + 3/2)}{\Gamma(K' + L' + 3/2)\Gamma(K+L+3/2)} \quad , \qquad (4.13)$$

where $P = 1/2(m + L + L')$. Note that by the orthonormality of the single particle, harmonic oscillator functions,

$$R(N'L,NL;\ 0) \ = \ \delta_{N'N} \quad .$$

First, we solve for the matrix element of the $\frac{1}{m} P_\lambda^2$ term in $3T_3$ of Eq. (4.5). This is a tensor of rank zero in the $\underset{\sim}{\lambda}$ variable and, hence, the reduced matrix element

$$\langle N_\lambda' L_\lambda' || P_\lambda^2 || N_\lambda L_\lambda \rangle \delta_{L_\lambda' L_\lambda} \quad ,$$

must be found in order to determine the matrix element of $\frac{1}{m} P_\lambda^2$ as specified by (4.12). By applying (11.13) and (7.30) to this reduced matrix element, we have

$$\langle N_\lambda' L_\lambda || P_\lambda^2 || N_\lambda L_\lambda \rangle \ = \ (-1)^{N_\lambda' + N_\lambda} \beta^4 \langle N_\lambda' L_\lambda || \lambda^2 || N_\lambda L_\lambda \rangle \quad ,$$

where β^4 is inserted to maintain the proper units. The recurrence relation (20)

$$xL_N^A(x) = -(N+1)L_{N+1}^A(x) + (2N+A+1)L_N^A(x) - (N+A)L_{N-1}^A(x) \quad , \quad (4.14)$$

as applied to a single particle, harmonic oscillator function in (9.3), yields,

$$r^2\beta^2|NLM\rangle = -\sqrt{(N+1)(N+L+3/2)}\ |N+1,LM\rangle + (2N+L+3/2)|NLM\rangle$$
$$-\sqrt{N(N+L+1/2)}\ |N-1,LM\rangle \quad . \quad (4.15)$$

Thus, one may easily show that

$$\langle N'_\lambda L_\lambda||P_\lambda^2||N_\lambda L_\lambda\rangle = \beta^2(\sqrt{(N_\lambda+1)(N_\lambda+L_\lambda+3/2)}\ \delta_{N'_\lambda,N_\lambda+1} + (2N_\lambda+L_\lambda+3/2)\delta_{N'_\lambda N_\lambda}$$
$$+ \sqrt{N_\lambda(N_\lambda+L_\lambda+1/2)}\ \delta_{N'_\lambda,N_\lambda-1}) \quad . \quad (4.16)$$

After we apply (7.33) to (4.12) to extract the reduced matrix element depending on the angular momentum coupled space variables $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ and then that in (4.16) depending on the $\underset{\sim}{\lambda}$ variable only, the matrix element of $\frac{1}{m}P_\lambda^2$ becomes

$$\frac{1}{m}(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{N'_\rho N_\rho}\delta_{L'_\lambda L_\lambda})\langle N'_\lambda L_\lambda||P_\lambda^2||N_\lambda L_\lambda\rangle$$

$$= \frac{\beta^2}{m}(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{N'_\rho N_\rho}\delta_{L'_\lambda L_\lambda})$$
$$\times(\sqrt{(N_\lambda+1)(N_\lambda+L_\lambda+3/2)}\ \delta_{N'_\lambda,N_\lambda+1} + (2N_\lambda+L_\lambda+3/2)\ \delta_{N'_\lambda N_\lambda}$$
$$+ \sqrt{N_\lambda(N_\lambda+L_\lambda+1/2)}\ \delta_{N'_\lambda,N_\lambda-1}) \quad . \quad (4.17)$$

The P_λ^4 term in $3T_3$ is processed in a similar manner and (4.15) is used to find $r^4\beta^4|NLM\rangle = r^2\beta^2(r^2\beta^2|NLM\rangle)$. The result is

$$
\begin{aligned}
&-\frac{\beta^4}{6m^3}\left(\delta_{\tau_{12}'\tau_{12}}\delta_{S'S}\delta_{S_{12}'S_{12}}\delta_{L'L}\delta_{L_\rho'L_\rho}\delta_{N_\rho'N_\rho}\delta_{L_\lambda'L_\lambda}\right)\\
&\quad\times\Big\{\sqrt{(N_\lambda+1)(N_\lambda+2)(N_\lambda+L_\lambda+3/2)(N_\lambda+L_\lambda+5/2)}\;\delta_{N_\lambda',N_\lambda+2}\\
&\quad+\sqrt{(N_\lambda+1)(N_\lambda+L_\lambda+3/2)}\;(4N_\lambda+2L_\lambda+5)\delta_{N_\lambda',N_\lambda+1}\\
&\quad+(6N_\lambda^2+L_\lambda^2+15/4+6N_\lambda L_\lambda+9N_\lambda+4L_\lambda)\delta_{N_\lambda'N_\lambda}+\sqrt{N_\lambda(N_\lambda+L_\lambda+1/2)}\\
&\quad\times(4N_\lambda+2L_\lambda+1)\delta_{N_\lambda',N_\lambda-1}+\sqrt{N_\lambda(N_\lambda-1)(N_\lambda+L_\lambda+1/2)(N_\lambda+L_\lambda-1/2)}\;\delta_{N_\lambda',N_\lambda-2}\Big\}\,.
\end{aligned}
\tag{4.18}
$$

The linear confinement term of (4.6) is a tensor of rank zero in the $\underset{\sim}{\rho}$ variable. As with the calculation of (4.17), (7.33) is used to extract the reduced matrix element depending on the coupled $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ variables and then that depending on $\underset{\sim}{\rho}$ only. This gives the matrix element of $3V_{12}^{LC}$ in (4.12) as

$$
\begin{aligned}
&2\sqrt{2}\;k\left(\delta_{\tau_{12}'\tau_{12}}\delta_{S'S}\delta_{S_{12}'S_{12}}\delta_{L'L}\delta_{L_\rho'L_\rho}\delta_{L_\lambda'L_\lambda}\delta_{N_\lambda'N_\lambda}\right)\langle N_\rho'L_\rho||\rho||N_\rho L_\rho\rangle\\
&\quad=\frac{2\sqrt{2}\;k}{\beta}\left(\delta_{\tau_{12}'\tau_{12}}\delta_{S'S}\delta_{S_{12}'S_{12}}\delta_{L'L}\delta_{L_\rho'L_\rho}\delta_{L_\lambda'L_\lambda}\delta_{N_\lambda'N_\lambda}\right)R(N_\rho'L_\rho,N_\rho L_\rho;\,1)\quad,
\end{aligned}
\tag{4.19}
$$

where (4.13) has been used to define the radial integration. Similarly, the matrix element of the nonrelativistic term in $3V_{12}^{OGE}$, Eq. (4.8), is found to be--

$$-\sqrt{2}\,\beta\alpha_s(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda})R(N'_\rho L_\rho, N_\rho L_\rho; -1) \quad . \quad (4.20)$$

Solving for the matrix element of the Darwin term of (4.8) is difficult and the operator needs to be rewritten before doing this. Using (7.29) as a guide, we may show that

$$\text{Darwin} = \frac{\alpha_s}{2\sqrt{2}\,m^2_\rho}\{-(P^2_\rho + \hat{\rho}\cdot(\hat{\rho}\cdot P_{\rho})P_{\rho}) + \frac{1}{3}[P_\lambda \times P_\lambda]_2 \cdot [\hat{\rho}\cdot\hat{\rho}]_2 + \frac{4}{9}(P^2_\lambda)\} . \quad (4.21)$$

This is a tensor of rank zero in the coupled λ and ρ variables. Hence, by applying (7.33), the matrix element of this term, as specified in (4.12), has the form

$$(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L})\langle(N'_\lambda L'_\lambda N'_\rho L'_\rho)L||\text{Darwin}||(N_\lambda L_\lambda N_\rho L_\rho)L\rangle \quad . \quad (4.22)$$

The first term in (4.21), which will be denoted D1, may be written as

$$D1 = -\frac{\alpha_s}{2\sqrt{2}\,m^2}(2\rho^{-1}P^2_\rho - \rho^{-3}L^2_\rho + 2\rho^{-2}\frac{\partial}{\partial\rho}) \quad . \quad (4.23)$$

This is a tensor of rank zero in the ρ variable. By applying (7.33) to (4.22), we find the matrix element of D1 to have the form

$$(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda})\langle N'_\rho L_\rho||D1||N_\rho L_\rho\rangle \quad . \quad (4.24)$$

Examining (4.16), we see that

$$P^2|NLM\rangle = \beta^2(\sqrt{(N+1)(N+L+3/2)}\,|N+1,LM\rangle + (2N+L+3/2)|NLM\rangle + \sqrt{N(N+L+1/2)}\,|N-1,LM\rangle) \quad . \quad (4.25)$$

Applying this and (4.13) to (4.24), we find the matrix element of the first term in D1 to be

$$-\frac{\alpha_s\beta^3}{\sqrt{2}\,m^2}(\delta_{\tau_{12}'\tau_{12}}\delta_{S'S}\delta_{S_{12}'S_{12}}\delta_{L'L}\delta_{L_\rho'L_\rho}\delta_{L_\lambda'L_\lambda}\delta_{N_\lambda'N_\lambda})\{\sqrt{(N_\rho+1)(N_\rho+L_\rho+3/2)}$$
$$\times R(N_\rho'L_\rho,N_\rho+1,L_\rho;\,-1) + (2N_\rho+L_\rho+3/2)R(N_\rho'L_\rho,N_\rho L_\rho;\,-1)$$
$$+\sqrt{N_\rho(N_\rho+L_\rho+1/2)}\,R(N_\rho'L_\rho,N_\rho-1,L_\rho;\,-1)\}\quad . \tag{4.26}$$

The matrix element of the second term in D1 is easily found by applying (7.4) and (4.13) to (4.24). The result is

$$\frac{\alpha_s\beta^3}{2\sqrt{2}\,m^2}(\delta_{\tau_{12}'\tau_{12}}\delta_{S'S}\delta_{S_{12}'S_{12}}\delta_{L'L}\delta_{L_\rho'L_\rho}\delta_{L_\lambda'L_\lambda}\delta_{N_\lambda'N_\lambda})L_\rho(L_\rho+1)R(N_\rho'L_\rho,N_\rho L_\rho;\,-3)\quad . \tag{4.27}$$

The matrix element of the third term in D1 is determined for two cases, $L_\rho > 0$ and $L_\rho = 0$. If $L_\rho > 0$, the differential relation (20)

$$x\frac{d}{dx}L_N^A(x) = NL_N^A(x) - (N+A)L_{N-1}^A(x) \tag{4.28}$$

and the recurrence relation in (4.14) may be used to write

$$\rho\frac{\partial}{\partial\rho}|N_\rho L_\rho M_\rho\rangle = 2y\frac{\partial}{\partial y}|N_\rho L_\rho M_\rho\rangle$$
$$= \sqrt{(N_\rho+1)(N_\rho+L_\rho+3/2)}\,|N_\rho+1,L_\rho M_\rho\rangle - 3/2\,|N_\rho L_\rho M_\rho\rangle$$
$$-\sqrt{N_\rho(N_\rho+L_\rho+1/2)}\,|N_\rho-1,L_\rho M_\rho\rangle\quad ,$$

where $y = \rho^2$. Using this and (4.13), we find the matrix element for $L_\rho > 0$ to be

$$-\frac{\alpha_s\beta^3}{\sqrt{2}\,m^2}\,(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda})\{\sqrt{(N_\rho+1)(N_\rho+L_\rho+3/2)}$$

$$\times R(N'_\rho L_\rho, N_\rho+1, L_\rho; -3) - 3/2\, R(N'_\rho L_\rho, N_\rho L_\rho; -3)$$

$$-\sqrt{N_\rho(N_\rho+L_\rho+1/2)}\,R(N'_\rho L_\rho, N_\rho-1, L_\rho; -3)\} \quad . \tag{4.29}$$

If $L_\rho = 0$, we must use a different method since $R(N'L,NL; -3)$ is undefined for $L = 0$. In this case, we operate with

$$\frac{1}{\rho}\,\frac{\partial}{\partial\rho} = 2\,\frac{\partial}{\partial y}$$

on the radial part of $|N_\rho L_\rho M_\rho\rangle$, where the angular part is $Y_{00}(\hat{\rho})$. The result is written as a sum over the radial parts of harmonic oscillator functions, multiplied by $Y_{00}(\hat{\rho})$, in order that (4.13) may be used. From this procedure, we find the matrix element for $L_\rho = 0$ as

$$-\frac{\alpha_s\beta^3}{\sqrt{2}\,m^2} = (\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda})\{-R(N'_\rho,0,N_\rho,0; -1)$$

$$+ 2[N_\rho!\,\Gamma(N_\rho+3/2)]^{1/2}\sum_{K=1}^{N_\rho}\sqrt{\Gamma(2K-1/2)}\,(-1)^K$$

$$\times\,[\Gamma(K+3/2)(K-1)!(N_\rho-K)!]^{-1}R(N'_\rho,0,0,2K-2; -1)\} \quad . \tag{4.30}$$

The second term in (4.21) is denoted D2. This operator is the scalar product of a second rank tensor in the $\underset{\sim}{\lambda}$ variable with one in the $\underset{\sim}{\rho}$ variable. Thus, from (7.34) applied to (4.22), we find

$$\langle(N'_\lambda L'_\lambda N'_\rho L'_\rho)L||D2||(N_\lambda L_\lambda N_\rho L_\rho)L\rangle$$

$$= \frac{\alpha_s}{6\sqrt{2}\,m^2}(-1)^{L'_\lambda+L_\rho-L}\hat{L}'_\lambda\hat{L}'_\rho\,W(L_\lambda L'_\lambda L_\rho L'_\rho;\,2L)$$

$$\times\langle N'_\lambda L'_\lambda||[P_{\underset{\sim}{\lambda}}\times P_{\underset{\sim}{\lambda}}]_2||N_\lambda L_\lambda\rangle\langle N'_\rho L'_\rho||\rho^{-1}[\hat{\rho}\times\hat{\rho}]_2||N_\rho L_\rho\rangle \quad , \qquad (4.31)$$

where $\hat{F} = \sqrt{2F+1}$ for a quantum number F. By using (11.13), we may write the $\underset{\sim}{\lambda}$-space reduced matrix element as

$$\langle N'_\lambda L'_\lambda||[P_{\underset{\sim}{\lambda}}\times P_{\underset{\sim}{\lambda}}]_2||N_\lambda L_\lambda\rangle = (-1)^{N'_\lambda+N_\lambda+L_\lambda}(i)^{L_\lambda+L'_\lambda}$$

$$\times\beta^4\langle N'_\lambda L'_\lambda||\lambda^2[\hat{\lambda}\times\hat{\lambda}]_2||N_\lambda L_\lambda\rangle \quad . \qquad (4.32)$$

From (7.22) and (7.15), we may show that

$$[\hat{r}\times\hat{r}]_{2m} = (8\pi/15)^{1/2}\,Y_{2m}(\hat{r}) \quad . \qquad (4.33)$$

Substituting this in (4.31) and (4.32), using (7.16) to perform the integrations over $\hat{\lambda}$ and $\hat{\rho}$, and using (4.13), we find the matrix element of D2 in (4.22) to be

$$\frac{\alpha_s\beta^3}{9\sqrt{2}\,m^2}(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L})\hat{L}_\lambda\hat{L}_\rho\,(-1)^{N'_\lambda+N_\lambda+1/2(L'_\lambda+L_\lambda)+L_\rho-L}$$

$$\times C(L_\lambda 2L'_\lambda;\,000)C(L_\rho 2L'_\rho;\,000)W(L_\lambda L'_\lambda L_\rho L'_\rho;\,2L)$$

$$\times R(N'_\lambda L'_\lambda, N_\lambda L_\lambda;\,2)R(N'_\rho L'_\rho, N_\rho L_\rho;\,-1) \quad . \qquad (4.34)$$

The third term in (4.21) is denoted D3. This operator is the product of a zeroth rank tensor in the $\underset{\sim}{\lambda}$ variable with one in the $\underset{\sim}{\rho}$ variable. From (7.34) applied to (4.22), the matrix element of D3 has the form

$$\frac{\sqrt{2}\,\alpha_s}{9m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\right)\langle N'_\lambda L_\lambda||P_\lambda^2||N_\lambda L_\lambda\rangle\langle N'_\rho L_\rho||\rho^{-1}||N_\rho L_\rho\rangle .$$

By applying (4.25) to the $\underset{\sim}{\lambda}$-space matrix element and (4.13) to the $\underset{\sim}{\rho}$-space matrix element, we find the matrix element of D3 to be

$$\frac{\sqrt{2}\,\alpha_s\beta^3}{9m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\right)R(N'_\rho L_\rho, N_\rho L_\rho; -1)$$
$$\times\left(\sqrt{(N_\lambda+1)(N_\lambda+L_\lambda+3/2)}\,\delta_{N'_\lambda,N_\lambda+1} + (2N_\lambda+L_\lambda+3/2)\delta_{N'_\lambda N_\lambda}\right.$$
$$\left. + \sqrt{N_\lambda(N_\lambda+L_\lambda+1/2)}\,\delta_{N'_\lambda,N_\lambda-1}\right) \quad . \tag{4.35}$$

The sum of (4.26), (4.27), (4.29) for $L_\rho > 0$, (4.30) for $L_\rho = 0$, (4.34), and (4.35) gives the full matrix element of the Darwin term.

The Fermi-contact term of (4.9) is the product of a zeroth rank tensor in the $\underset{\sim}{\rho}$ variable with one in the $\underset{\sim}{S}_{12}$ variable. From (7.34) and (7.33), we find the matrix element to have the form

$$\frac{\pi\alpha_s}{\sqrt{2}\,m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda}\right)$$
$$\times\langle N'_\rho L_\rho||\delta^3(\underset{\sim}{\rho})||N_\rho L_\rho\rangle\langle S_{12}||1+\frac{8}{3}\,\underset{\sim}{s}_1\cdot\underset{\sim}{s}_2||S_{12}\rangle \quad , \tag{4.36}$$

where (7.13) has been used to reverse the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ angular momentum coupling. Because $\underset{\sim}{\rho} = 0$ in the matrix element, the ρ^{L_ρ} factor in $|N_\rho L_\rho M_\rho\rangle$ in (9.3) implies that $\langle N'_\rho L_\rho||\delta^3(\underset{\sim}{\rho})||N_\rho L_\rho\rangle$ is zero unless $L_\rho = 0$. In spherical coordinates (21),

$$\delta^3(\underset{\sim}{\rho}) = \frac{\delta(\rho)}{\rho^2}\sum_{LM} Y^*_{LM}(0,0)Y_{LM}(\theta_\rho,\phi_\rho) \tag{4.37}$$

Hence, the angular momentum constraints imply that if $L_\rho = 0$ in (4.36), then $L = 0$ in (4.37). From (9.3) and (9.4), we find the $\underset{\sim}{\rho}$-space matrix element to be

$$\langle N'_\rho L_\rho || \delta^3(\underset{\sim}{\rho}) || N_\rho L_\rho \rangle = \delta_{L_\rho,0}\, 2\beta^3\pi^{-2} \left(\frac{\Gamma(N'_\rho+3/2)\Gamma(N_\rho+3/2)}{N'_\rho! N_\rho!} \right)^{1/2} . \quad (4.38)$$

Using (3.7), one may show that

$$\underset{\sim}{s}_1 \cdot \underset{\sim}{s}_2 = 1/2(\underset{\sim}{s}_{12}^2 - \underset{\sim}{s}_1^2 - \underset{\sim}{s}_2^2) .$$

Substituting this and applying (7.4), we find the $\underset{\sim}{S}_{12}$-space matrix element to be

$$\langle S_{12} || 1 + \frac{8}{3}\, \underset{\sim}{s}_1 \cdot \underset{\sim}{s}_2 || S_{12} \rangle = -\delta_{S_{12},0} + \frac{5}{3}\, \delta_{S_{12},1} . \quad (4.39)$$

By substituting (4.38) and (4.39) into (4.36), the matrix element of the Fermi-contact term is determined to be

$$\frac{\sqrt{2}\, \alpha_S \beta^3}{\pi m^2} (\delta_{\tau'_{12}\tau_{12}} \delta_{S'S} \delta_{S'_{12}S_{12}} \delta_{L'L} \delta_{L'_\rho L_\rho} \delta_{L'_\rho,0} \delta_{L'_\lambda L_\lambda} \delta_{N'_\lambda N_\lambda})$$

$$\times \left(\frac{\Gamma(N'_\rho+3/2)\Gamma(N_\rho+3/2)}{N'_\rho! N_\rho!} \right)^{1/2} (-\delta_{S_{12},0} + \frac{5}{3}\, \delta_{S_{12},1}) . \quad (4.40)$$

Let us denote the first term in the spin-orbit operator of (4.10) SO1; that is

$$SO1 = \frac{3\alpha_S}{2\sqrt{2}\, m^2} (\rho^{-3} \underset{\sim}{L}_\rho \cdot \underset{\sim}{S}_{12}) .$$

This is the scalar product of a first rank tensor in the $\underset{\sim}{\rho}$ variable with one in the $\underset{\sim}{S}_{12}$ variable. Equations (7.34) and (7.33) then set the form of the matrix element of S01 as

$$\frac{3\alpha_S}{2\sqrt{2}\,m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda}\right)(-1)^{L_\lambda+L'_\rho-L+L'+S+S'+S_{12}-J+3/2}$$

$$\times\, L'_\rho LL'SS'S_{12}\, W(LL'SS';\,1J)W(L_\rho L'_\rho LL';\,1L_\lambda)W(S_{12}S'_{12}SS';\,1,1/2)$$

$$\times\, \langle N'_\rho L'_\rho||\rho^{-3}\underset{\sim}{L}_\rho||N_\rho L_\rho\rangle\langle S'_{12}||\underset{\sim}{S}_{12}||S_{12}\rangle \quad , \tag{4.41}$$

where (7.13) has been used to reverse the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ coupling. By applying (7.31) to each reduced matrix element in (4.41) and using (4.13), we find the matrix element of S01 to be

$$\frac{3\sqrt{3}\,\alpha_S\beta^3}{2m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{S'_{12}S_{12}}\delta_{S_{12},1}\delta_{L'_\rho L_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda}\right)(-1)^{L_\lambda+L_\rho-L+L'+S+S'-J+1/2}$$

$$\times\, \sqrt{L_\rho(L_\rho+1)}\,\hat{L}_\rho\hat{L}\hat{L}'\hat{S}\hat{S}'\, W(L_\rho L_\rho LL';\,1L_\lambda)W(11SS';\,1,1/2)W(LL'SS';\,1J)$$

$$\times\, R(N'_\rho L_\rho, N_\rho L_\rho;\,-3) \quad . \tag{4.42}$$

The second term in (4.10) is denoted S02 and this may be written in the form

$$S02 = \frac{i\alpha_S}{2\sqrt{3}\,m^2}\left(\rho^{-2}[\underset{\sim}{P}_\lambda \times \hat{\rho}]_1 \cdot [\underset{\sim}{s}_2 - \underset{\sim}{s}_1]_1\right)$$

by using (7.28) to write the cross product as a coupled tensor. From (7.34), (7.32), and (7.33), we find the matrix element of S02 to have the form

$$\frac{i\alpha_s}{2m^2}(\delta_{\tau'_{12}\tau_{12}})\,(-1)^{L'+S+S'+S_{12}-J+1/2}\,\hat{L}'_\lambda\hat{L}'_\rho\hat{L}\hat{L}'\hat{S}\hat{S}'\hat{S}'_{12}\,W(LL'SS';\,1J)$$

$$\times\,W(S_{12}S'_{12}SS';\,1,1/2)\begin{Bmatrix} L_\lambda & L_\rho & L \\ 1 & 1 & 1 \\ L'_\lambda & L'_\rho & L' \end{Bmatrix}$$

$$\times\,\langle N'_\lambda L'_\lambda||P_{\underset{\sim}{\lambda}}||N_\lambda L_\lambda\rangle\langle N'_\rho L'_\rho||\rho^{-2}\hat{\rho}||N_\rho L_\rho\rangle\langle S'_{12}||s_{\underset{\sim}{2}}-s_{\underset{\sim}{1}}||S_{12}\rangle \quad . \quad (4.43)$$

By using (11.13), we may write the $\underset{\sim}{\lambda}$-space matrix element as

$$\langle N'_\lambda L'_\lambda||P_{\underset{\sim}{\lambda}}||N_\lambda L_\lambda\rangle = (-1)^{N_\lambda+N'_\lambda+L_\lambda}\,(i)^{L_\lambda+L'_\lambda}\,\beta^2\langle N'_\lambda L'_\lambda||\lambda\hat{\lambda}||N_\lambda L_\lambda\rangle \quad . \quad (4.44)$$

Then we use (7.22), (7.15), and the two recurrence relations (20)

$$L_N^A = L_N^{A+1} - L_{N-1}^{A+1} \quad (4.45)$$

and

$$xL_N^A(x) = (N+A)L_N^{A-1}(x) - (N+1)L_{N+1}^{A-1}(x) \quad (4.46)$$

on the right hand side of (4.44) to show that

$$\begin{aligned}\langle N'_\lambda L'_\lambda||P_{\underset{\sim}{\lambda}}||N_\lambda L_\lambda\rangle = \frac{i\beta}{\hat{L}'_\lambda}\,&(\sqrt{(L_\lambda+1)(N_\lambda+L_\lambda+3/2)}\,\delta_{N'_\lambda N_\lambda}\delta_{L'_\lambda,L_\lambda+1} \\ &+\sqrt{(L_\lambda+1)N_\lambda}\,\delta_{N'_\lambda,N_\lambda-1}\delta_{L'_\lambda,L_\lambda+1}+\sqrt{L_\lambda(N_\lambda+1)}\,\delta_{N'_\lambda,N_\lambda+1} \\ &\times\,\delta_{L'_\lambda,L_\lambda-1}+\sqrt{L_\lambda(N_\lambda+L_\lambda+1/2)}\,\delta_{N'_\lambda N_\lambda}\delta_{L'_\lambda,L_\lambda-1}) \quad . \quad (4.47)\end{aligned}$$

By using (7.22), (7.16), and (4.13), we find the $\underset{\sim}{\rho}$-space matrix element to be

$$\langle N'_\rho L'_\rho || \rho^{-2} \hat{\rho} || N_\rho L_\rho \rangle = \beta^2 \frac{\hat{L}_\rho}{\hat{L}'_\rho} C(L_\rho 1 L'_\rho; 000)\, R(N'_\rho L'_\rho, N_\rho L_\rho; -2) \quad . \tag{4.48}$$

Then, applying (7.33) to the S_{12}-space matrix element and using (7.13) to reverse the s_1 and s_2 coupling in $\langle S'_{12} || s_2 || S_{12} \rangle$, we find

$$\langle S'_{12} || s_2 - s_1 || S_{12} \rangle = (-1)^{S_{12}+1} \hat{S}_{12}\, \delta_{S_{12}+S'_{12},1} \quad . \tag{4.49}$$

Substituting (4.47), (4.48), and (4.49) into (4.43), we find the matrix element of SO2 to be

$$\begin{aligned} &\frac{\sqrt{3}\,\alpha_s \beta^3}{2m^2} (\delta_{\tau'_{12}\tau_{12}} \delta_{S_{12}+S'_{12},1})\, (-1)^{L'+(S+S'+1/2-J)}\, \hat{L}_\rho \hat{L} \hat{L}' \hat{S} \hat{S}'\, C(L_\rho 1 L'_\rho; 000) \\ &\quad \times W(LL'SS'; 1J) W(S_{12} S'_{12} SS'; 1, 1/2) \begin{Bmatrix} L_\lambda & L_\rho & L \\ 1 & 1 & 1 \\ L'_\lambda & L'_\rho & L' \end{Bmatrix} R(N'_\rho L'_\rho, N_\rho L_\rho; -2) \\ &\quad \times \left(\sqrt{(L_\lambda+1)(N_\lambda+L_\lambda+3/2)}\, \delta_{N'_\lambda N_\lambda} \delta_{L'_\lambda, L_\lambda+1} + \sqrt{(L_\lambda+1)N_\lambda}\, \delta_{N'_\lambda, N_\lambda-1} \delta_{L'_\lambda, L_\lambda+1} \right. \\ &\quad \left. + \sqrt{L_\lambda (N_\lambda+1)}\, \delta_{N'_\lambda, N_\lambda+1} \delta_{L'_\lambda, L_\lambda-1} + \sqrt{L_\lambda (N_\lambda+L_\lambda+1/2)}\, \delta_{N'_\lambda N_\lambda} \delta_{L'_\lambda, L_\lambda-1} \right) \quad . \end{aligned} \tag{4.50}$$

Then, the full matrix element of the spin-orbit term is the sum of (4.42) and (4.50).

The last term in H is the tensor-force operator of (4.12), which we may write in coupled tensor notation as

$$\text{tensor-force} = \frac{3\alpha_s}{\sqrt{2}\, m^2} (\rho^{-3} [\hat{\rho} \times \hat{\rho}]_2 \cdot [s_1 \times s_2]_2) \tag{4.51}$$

by using (7.29). This is the scalar product of a second rank tensor in the $\underset{\sim}{\rho}$ variable with one in the $\underset{\sim}{S}_{12}$ variable. Hence, by applying (7.34) and (7.33), we find the matrix element of the tensor-force operator to have the form

$$\frac{3\alpha_s}{\sqrt{2}\,m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda}\right)(-1)^{L_\lambda+L'_\rho-L+L'+S+S'+S_{12}-J-1/2}\hat{L}'_\rho\hat{L}\hat{L}'\hat{S}\hat{S}'\hat{S}'_{12}$$

$$\times W(LL'SS';\,2J)W(L_\rho L'_\rho LL';\,2L_\lambda)W(S_{12}S'_{12}SS';\,2,1/2)$$

$$\times\langle N'_\rho L'_\rho||\rho^{-3}[\hat{\rho}\times\hat{\rho}]_2||N_\rho L_\rho\rangle\langle S'_{12}||[\underset{\sim}{s}_1\times\underset{\sim}{s}_2]_2||S_{12}\rangle \quad , \tag{4.52}$$

where (7.13) has been used to reverse the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ coupling. By applying (4.33), (7.16), and (4.13) to the $\underset{\sim}{\rho}$-space matrix element, we find

$$\langle N'_\rho L'_\rho||\rho^{-3}[\hat{\rho}\times\hat{\rho}]_2||N_\rho L_\rho\rangle = \beta^3\sqrt{\frac{2}{3}}\,\frac{\hat{L}_\rho}{\hat{L}'_\rho}\,C(L_\rho 2L'_\rho;\,000)R(N'_\rho L'_\rho,N_\rho L_\rho;\,-3) \quad . \tag{4.53}$$

Equation (7.32) applied to the $\underset{\sim}{S}_{12}$-space matrix element yields

$$\langle S'_{12}||[\underset{\sim}{s}_1\times\underset{\sim}{s}_2]_2||S_{12}\rangle = \frac{\sqrt{15}}{6}\,\delta_{S'_{12}S_{12}}\delta_{S_{12},1} \quad . \tag{4.54}$$

Substituting (4.53) and (4.54) into (4.52), we find the matrix element of the tensor-force term to be

$$\frac{\sqrt{15}\,\alpha_s\beta^3}{2m^2}\left(\delta_{\tau'_{12}\tau_{12}}\delta_{S'_{12}S_{12}}\delta_{S_{12},1}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda}\right)(-1)^{L_\lambda+L_\rho+L+L'+S+S'-J+1/2}$$

$$\times\hat{L}_\rho\hat{L}\hat{L}'\hat{S}\hat{S}'\,C(L_\rho 2L'_\rho;\,000)W(LL'SS';\,2J)W(L_\rho L'_\rho LL';\,2L_\lambda)$$

$$\times W(11SS';\,2,1/2)R(N'_\rho L'_\rho,N_\rho L_\rho;\,-3) \quad . \tag{4.55}$$

The matrix element of H in (4.13) is the sum of (4.17)-(4.20), (4.26)-(4.27), (4.29) for $L_\rho > 0$, (4.30) for $L_\rho = 0$, (4.34)-(4.35), (4.40), (4.42), (4.50), and (4.55). The matrix element of H between two basis functions is the sum over these more fundamental matrix elements as specified by (4.2).

B. Matrix Elements of the Magnetic Moment and Charge Radius Operators

After diagonalizing H, the magnetic moments and root mean square charge radii of the different charge states of the lowest lying N $1/2^+$ and Δ $3/2^+$ levels will be determined. If $|JM,\tau M_\tau\rangle$ is the eigenvector of an N or Δ J^π resonance, then the resonance's magnetic moment is defined as

$$\mu = \langle JJ,\tau M_\tau | \sum_i \mu_i (\ell_{i,0} + s_{i,0}) | JJ,\tau M_\tau \rangle \quad .$$

As assumed in Chapter II, $\mu_i = eQ_i/2m$ is the intrinsic magnetic moment of the ith quark. Here, $\ell_{i,0}$ and $s_{i,0}$ are the z-component of the ith quark's orbital and spin angular momentum, respectively. Magnetic moments are measured in units of nuclear magnetons,

$$\mu_N = e/2M_p \quad ,$$

where

$$M_p = 938.2 \text{ MeV}$$

is the p mass. Thus, in units of μ_N,

$$\mu = \langle JJ,\tau M_\tau | \frac{M_P}{m} \sum_i Q_i (\ell_{i,0} + 2s_{i,0}) | JJ,\tau M_\tau \rangle \quad . \tag{4.56}$$

The mean square charge radius is defined as

$$r_Q^2 = \langle JM,\tau M_\tau | \sum_i Q_i \, r_i^2 | JM,\tau M_\tau \rangle \tag{4.57}$$

and the root mean square (rms) charge radius is then

$$r_Q = \sqrt{|r_Q^2|} \quad , \tag{4.58}$$

where the sign of r_Q is taken to be the sign of r_Q^2. In terms of isospin operators, we may write

$$Q_i = (\tau_{i,0} + 1/6) \quad , \tag{4.59}$$

where $\tau_{i,0}$ is the z-component of the ith quark's isospin, by using Table 1.1 for the u and d quarks. Substituting (4.59) into (4.56) and (4.57), invoking the symmetry property of the basis states used in deriving (4.4), and using the Jacobi coordinates of Table 3.4, we find

$$\mu = \langle JJ,\tau M_\tau | \frac{M_P}{m} (3\tau_{3,0} + \tfrac{1}{2})(\tfrac{2}{3} L_{\lambda,0} + 2s_{3,0}) | JJ,\tau M_\tau \rangle \tag{4.60}$$

and

$$r_Q^2 = \langle JM,\tau M_\tau | (3\tau_{3,0} + \tfrac{1}{2})(\tfrac{2}{3}\lambda^2) | JM,\tau M_\tau \rangle \quad . \tag{4.61}$$

The eigenvector of a resonance is given generally as

$$|JM,\tau M_\tau\rangle = \sum_n e_n |\psi_n\rangle \quad ,$$

where the $|\psi_n\rangle$ are the basis states in an N or Δ J^π set of Table 10.2. Thus, the expectation value of an operator $\hat{0}$ is

$$\langle JM,\tau M_\tau|\hat{0}|JM,\tau M_\tau\rangle = \sum_{mn} e_m^* e_n \langle\psi_m|\hat{0}|\psi_n\rangle \quad , \tag{4.62}$$

where $\langle\psi_m|\hat{0}|\psi_n\rangle$ is given by (4.2). By determining the fundamental matrix elements of $\hat{0}$ on the right hand side of (4.2) and performing the summations specified in (4.2) and (4.62), we find the expectation value of $\hat{0}$. Hence, we need to determine the magnetic moment matrix element,

$$\langle(\tau'_{12})\tau M_\tau|\langle[L'S']JJ|\frac{M_p}{m}(3\tau_{3,0}+\tfrac{1}{2})(\tfrac{2}{3}L_{\lambda,0}+2s_{3,0})|[LS]JJ\rangle|(\tau_{12})\tau M_\tau\rangle \quad , \tag{4.63}$$

and the charge radius matrix element,

$$\langle(\tau'_{12})\tau M_\tau|\langle[L'S']JM|(3\tau_{3,0}+\tfrac{1}{2})(\tfrac{2}{3}\lambda^2)|[LS]JM\rangle|(\tau_{12})\tau M_\tau\rangle \quad . \tag{4.64}$$

Before doing this, we must find the isospin matrix element

$$\langle(\tau'_{12})\tau',M'_\tau|(3\tau_{3,0}+1/2)|(\tau_{12})\tau,M_\tau\rangle$$

$$= \delta_{M'_\tau M_\tau}[3C(\tau 1\tau';\, M_\tau 0 M_\tau)\langle(\tau_{12})'\tau||\tau_3||(\tau_{12})\tau\rangle + \tfrac{1}{2}\delta_{\tau'_{12}\tau_{12}}\delta_{\tau'\tau}] \quad . \tag{4.65}$$

Inverting the coupling of $\underset{\sim}{\tau}_{12}$ and $\underset{\sim}{\tau}_3$ by using (7.13) and applying (7.33), we find

$$\langle(\tau'_{12})\tau',M'_\tau|(3\tau_{3,0}+1/2)|(\tau_{12})\tau,M_\tau\rangle$$

$$= \frac{1}{2}\delta_{\tau'_{12}\tau_{12}}\delta_{M'_\tau M_\tau}[(-1)^{\tau_{12}+\tau-1/2}\,3\sqrt{6}\,\hat{\tau}C(\tau 1\tau';\,M_\tau 0 M_\tau)$$

$$\times W(1/2,1/2,\tau\tau';\,1\tau_{12}) + \delta_{\tau\tau'}] \quad , \tag{4.66}$$

where (7.31) has also been used. The matrix element in (4.66) will be denoted $T(\tau_{12},M_\tau;\,\tau',\tau)$. In Table 4.1, a list of the values of T needed in this thesis is given. Note that $T(\tau_{12},M_\tau;\,\tau,\tau') = T(\tau_{12},M_\tau;\,\tau',\tau)$. Using this definition of T and applying (7.30) and (7.33) to (4.63), we find the magnetic moment matrix element to be

$$\frac{M_P}{m}T(\tau_{12},M_\tau;\,\tau,\tau)(\delta_{S'_{12}S_{12}}\delta_{L'_\rho L_\rho}\delta_{N'_\rho N_\rho}\delta_{L'_\lambda L_\lambda}\delta_{N'_\lambda N_\lambda})[\delta_{S'S}\,\frac{2}{3}\sqrt{\frac{J}{J+1}}\sqrt{L_\lambda(L_\lambda+1)}$$

$$\times \hat{L}_\lambda\hat{L}\hat{L}'\hat{J}\,(-1)^{L_\lambda+L_\rho+L+L'+S-J}\,W(LL'JJ;\,1S)W(L_\lambda L_\lambda LL';\,1L_\rho)$$

$$+ \delta_{L'L}\sqrt{6}\sqrt{\frac{J}{J+1}}\,\hat{S}\hat{S}'\hat{J}\,(-1)^{L+S_{12}+S+S'+J+1/2}$$

$$\times W(SS'JJ;\,1L)W(1/2,1/2,SS';\,1S_{12})] \quad , \tag{4.67}$$

where (7.31) has been used to determine $\langle N'_\lambda L'_\lambda||L_\lambda||N_\lambda L_\lambda\rangle$ and $\langle s_3||s_3||s_3\rangle$ and (7.13) was used to reverse the angular momentum couplings in the matrix element depending on the spin variables. By applying (7.30) and (7.33) to (4.64), we find the matrix element of the charge radius operator to be

Table 4.1. Values of $T(\tau_{12},M_\tau;\ \tau',\tau)$, where T is defined by Eq. (4.66)

τ_{12}	M_τ	τ'	τ	T
1	1/2	1/2	1/2	0
1	1/2	3/2	1/2	$-\sqrt{2}$
0	1/2	1/2	1/2	2
1	-1/2	1/2	1/2	1
1	-1/2	3/2	1/2	$-\sqrt{2}$
0	-1/2	1/2	1/2	-1
1	3/2	3/2	3/2	2
1	1/2	3/2	3/2	1
1	-1/2	3/2	3/2	0
1	-3/2	3/2	3/2	-1

$$\frac{2}{3\beta^2}\, T(\tau_{12},M_\tau;\ \tau,\tau)(\delta_{S'S}\delta_{S'_{12}S_{12}}\delta_{L'L}\delta_{L'_\rho L_\rho}\delta_{N'_\rho N_\rho}\delta_{L'_\lambda L_\lambda})\,[-\sqrt{(N_\lambda+1)(N_\lambda+L_\lambda+3/2)}$$

$$\times\ \delta_{N'_\lambda,N_\lambda+1} + (2N_\lambda + L_\lambda + 3/2)\delta_{N'_\lambda N_\lambda} - \sqrt{N_\lambda(N_\lambda+L_\lambda+1/2)}\ \delta_{N'_\lambda,N_\lambda-1}] \quad , \tag{4.68}$$

where (4.15) has been used to determine $\langle N'_\lambda L_\lambda||\lambda^2||N_\lambda L_\lambda\rangle$. Thus, (4.67) and (4.68) used with (4.2) and (4.62) give the magnetic moment and mean square charge radius, respectively.

C. Diagonalization of the Hamiltonian and Results

Before diagonalizing H, we perform a few, simple calculations to give a rough idea of what the parameters, m, α_s, k, and β, may be for the 6ω basis and how they may be determined. Let us restrict ourselves to a 0ω basis. By Table 10.2, this means that only the state $|\psi_N\rangle = [1\rangle$ in the N $1/2^+$ basis and the state $|\psi_\Delta\rangle = [1\rangle$ in the Δ $3/2^+$ basis are used. In these states, the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ oscillators are in the ground state. In this basis, the N(939) $1/2^+$ and $\Delta(1232)$ $3/2^+$ are the only resonances which can be described. We define the resonant masses as

$$939 = \langle\psi_N|H|\psi_N\rangle$$

and

$$1232 = \langle\psi_\Delta|H|\psi_\Delta\rangle \quad .$$

Then by using the previously derived matrix elements, we find that

$$939 = 3m + 1.5\,\beta^2 m^{-1} - 0.625\,\beta^4 m^{-3} + 3.192\,k\beta^{-1} - 1.596\,\alpha_s\beta$$

$$- 0.3989\,\alpha_s\beta^3 m^{-2} \tag{4.69}$$

and

$$1232 = 3m + 1.5\,\beta^2 m^{-1} - 0.625\,\beta^4 m^{-3} + 3.192\,k\beta^{-1} - 1.596\,\alpha_s\beta + 0.1330\,\alpha_s\beta^3 m^{-2} \quad . \tag{4.70}$$

Also, the p rms charge radius is

$$r_{Q-p} = \beta^{-1} \tag{4.71}$$

and the p magnetic moment is

$$\mu_p = \frac{M_p}{m} = \frac{938.2}{m} \quad . \tag{4.72}$$

Note that in the notation of (4.1),

$$|\psi_N\rangle = 1/\sqrt{2}[|[(00,00)0(1)1/2]1/2,M\rangle|(1)1/2,M_\tau\rangle + |[(00,00)0(0)1/2]1/2,M\rangle|(0)1/2,M_\tau\rangle]$$

and

$$|\psi_\Delta\rangle = |[(00,00)0(1)3/2]3/2,M\rangle|(1)3/2,M_\tau\rangle \quad ,$$

where $M_\tau = 1/2$ in $|\psi_N\rangle$ for the p. Experimentally, $r_{Q-p} = 0.88$ F = $(224\ \text{MeV})^{-1}$, where $\hbar c = 197$ MeV - F = 1, and $\mu_p = 2.793$. Fitting the theoretical values of (4.69)-(4.72) exactly with the experimental values gives

$$m = 336\ \text{MeV} \quad ,$$

$$\beta = 224\ \text{MeV} \quad ,$$

$$\alpha_s = 5.53 \quad ,$$

and

$$k = 1.37 \times 10^5 \text{ MeV}^2 \quad .$$

In this calculation, μ_p fixed the quark mass, r_{Q-p} fixed the inverse length parameter, the mass difference $\Delta(1232)$ - N(939) fixed α_s, and then N(939) fixed k. These parameters then predict that for the n, with $M_\tau = -1/2$,

$$\mu_n = -2/3(M_p/m) = -1.862$$

and

$$r_{Q-n} = 0 \quad ,$$

and experimentally, $\mu_n = -1.913$ and $r_{Q-n} = -0.336$ F.

The sum of a particle's rest mass and kinetic energy is related to its velocity, v, by

$$m + E_k = m(1 - v^2)^{-1/2} \quad .$$

Solving for v^2, we have

$$v^2 = 1 - (1 + E_k/m)^{-2} \quad . \tag{4.73}$$

From (4.69) and (4.70), the quark kinetic energy is

$$E_k = \frac{1}{3}(1.5\,\beta^2 m^{-1} - 0.625\,\beta^4 m^{-3}) = 61 \text{ MeV} \quad .$$

Substituting this in (4.73), we find

$$v^2 = 0.28$$

in this 0ω basis. Note that once the quark mass is determined, E_k is a function of β. Since β is fixed by r_{Q-p}, r_{Q-p} basically determines the quark velocities.

Fixing the parameters as was done in this simple calculation is not possible for the 2ω, 4ω, and 6ω bases since the exact form of an eigenvector is not known until after diagonalizing H and H cannot be diagonalized numerically until the parameters are given. Hence, we fix the parameters by performing a chi-square minimization in fitting theoretical results to a small set of experimental data. Thus, if E_i is the ith experimental data point, T_i the corresponding theoretical prediction, and w_i a weight factor, then we minimize

$$\chi^2 = \sum_i w_i(1 - T_i/E_i)^2 \tag{4.74}$$

by varying the four parameters, where $\sum_i w_i = 1$. The experimental data used are the three lowest N $1/2^+$ levels with masses of 939, 1470, and 1710, the two lowest Δ $3/2^+$ levels with masses of 1232 and 1690, r_{Q-p}, μ_p, and μ_n. From the results of the 0ω calculation, we expect that, to a large extent, r_{Q-p} will determine and scale the kinetic energies, μ_p and μ_n will determine m, the difference Δ(1232) - N(939) will determine α_s, and N(939) will determine k. To a lesser extent, the difference N(1710) - N(1470) will determine α_s and the differences N(1710) - N(939) and Δ(1690) - Δ(1232) will determine k.

Using the same relative weights for the data in each basis, the parameters and theoretical predictions in Table 4.2 result. The value of χ^2 in (4.74), the average, relative error between the eight

Table 4.2. The results of fixing the parameters through a χ^2 minimization in the 2ω, 4ω, and 6ω bases

Experimental Data		Relative Weights	Theoretical Results		
			2ω	4ω	6ω
$N\ 1/2^+$	939 MeV	10	963	957	954
	1470 MeV	1	1624	1600	1583
	1710 MeV	1	1693	1693	1689
$\Delta\ 3/2^+$	1232 MeV	7	1141	1158	1168
	1690 MeV	1	1758	1776	1781
r_{Q-p} = 0.88 F		2	0.794	0.797	0.801
μ_p = 2.793 μ_N		2	2.903	2.909	2.908
μ_n = -1.913 μ_N		2	-1.932	-1.934	-1.932
χ^2			0.0031	0.0023	0.0019
Ave. Error			5.0%	4.7%	4.4%
m (MeV)			321.8	320.4	320.1
α_S			2.008	2.098	2.130
β (MeV)			207.3	183.1	165.8
k (10^4 MeV^2)			5.785	6.450	6.779
ω (MeV)			134	105	86

experimental and theoretical values, and the oscillator frequency,

$$\omega = \beta^2/m \quad ,$$

are also given. Note that χ^2 and the average error decrease as the basis size increases. (The N $1/2^+$ basis consists of 5, 15, and 34 states for the 2ω, 4ω, and 6ω bases, respectively, and the Δ $3/2^+$ basis consists of 4, 12, and 28 states.) This indicates that H may contain the major physical properties reflected in the experimental data since we expect to get better approximations to the exact eigenvalues and eigenvectors as the basis size increases. Also, the relative weights assigned to the data points merely reflect our personal choice of which theoretical results should more closely match the corresponding experimental data.

We then diagonalize H in the 2ω, 4ω, and 6ω bases using the corresponding sets of parameters in Table 4.2. The lowest lying eigenvalues, up to a maximum of four, are given in Table 4.3 and are compared with experimental masses where possible. The numbers in parentheses beside the experimental masses are the experimental widths; unknown widths are labeled with a question mark. The stars rate the confidence in the experimental masses on a rising scale from one to four. The 6ω results are the theoretical predictions of the model since these are considered to be the best approximations to the exact eigenvalues of H. An examination of the 6ω results shows a respectable agreement between theory and experiment. There are some notable exceptions, however, in comparing the theoretical results to the well-known, three and four star levels. The two, lowest lying, N $1/2^-$ levels differ from experiment by

Table 4.3. Convergence of the mass eigenvalues of H using 2ω, 4ω, and 6ω bases and comparison with experimental masses. Masses and widths are given in units of MeV

	2ω Basis	4ω Basis	6ω Basis	Exp(width)	Confidence
N 1/2+	963	957	954	939(0)	****
	1624	1600	1583	1470(200)	****
	1693	1693	1689	1710(120)	****
	1806	1820	1822		
N 1/2-	1451	1396	1357	1535(150)	****
	1540	1510	1486	1650(150)	****
		2154	2080	2100(?)	*
		2223	2149		
N 3/2+	1692	1684	1676	1540(?)	*
	1724	1727	1725	1810(200)	****
	1741	1754	1757		
	1796	1816	1823		
N 3/2-	1568	1549	1531	1520(125)	****
	1602	1590	1577	1700(120)	****
		2181	2113	2040(?)	**
		2249	2183		
N 5/2+	1777	1791	1794	1688(130)	****
	1814	1837	1845	2000(?)	**
	1860	1894	1907		
		2395	2353		
N 5/2-	1675	1685	1685	1670(155)	****
		2266	2202	2100(?)	**
		2286	2228		
		2316	2267		
N 7/2+	1918	1962	1982	1990(250)	***
		2444	2403		
		2462	2428		
		2472	2442		
N 7/2-		2322	2275	2190(250)	****
		2336	2296		
		2388	2346		
		2405	2367		
N 9/2+		2485	2452	2220(300)	****
		2523	2500		
		2546	2527		
		2597	2574		
N 9/2-		2381	2351	2200(250)	****
			2932	2800(?)	*
			2951		
			2971		
N 11/2+		2577	2563		
		2660	2651		
			3091		
			3109		

Table 4.3. Continued

	2ω Basis	4ω Basis	6ω Basis	Exp(width)	Confidence
N 11/2-			2965	2600(400)	***
			2980		
			3004		
			3043		
N 13/2+			3116	2700(?)	*
			3147		
			3159		
			3211		
N 13/2-			3008		
			3141		
N 15/2+			3179		
			3256		
Δ 1/2+	1722	1732	1733	1550(?)	**
	1741	1758	1763	1910(220)	****
		2376	2335		
		2470	2430		
Δ 1/2-	1595	1584	1571	1650(140)	****
		2208	2151	1900(?)	**
		2293	2241		
		2483	2452		
Δ 3/2+	1141	1158	1168	1232(115)	****
	1758	1776	1781	1690(250)	***
	1811	1834	1842	1960(?)	**
	1905	1961	1985		
Δ 3/2-	1618	1614	1605	1670(200)	****
		2220	2166		
		2292	2233		
		2326	2278		
Δ 5/2+	1813	1843	1854	1890(250)	****
	1842	1871	1883		
		2432	2400		
		2451	2419		
Δ 5/2-		2258	2212	1960(200)	***
		2311	2262		
		2344	2299		
		2410	2368		
Δ 7/2+	1869	1911	1929	1950(240)	****
		2478	2450		
		2519	2494		
		2555	2527		
Δ 7/2-		2345	2307		
		2390	2352		
			2905		
			2935		

Table 4.3. Continued

	2ω Basis	4ω Basis	6ω Basis	Exp(width)	Confidence
Δ 9/2+		2513	2492	2300(?)	*
		2531	2508		
		2610	2591		
			3086		
Δ 9/2-		2439	2409	2500(?)	*
			2928		
			2959		
			2988		
Δ 11/2+		2549	2535	2420(300)	***
			3115		
			3143		
			3172		
Δ 11/2-			2984		
			3012		
			3105		
			3144		
Δ 13/2+			3138		
			3151		
			3218		
			3291		
Δ 13/2-			3037	2750(?)	*
Δ 15/2+			3163	2950(?)	*
			3329		

about 10% and are converging in the wrong direction. The lowest N $11/2^-$ level differs by 14% and the lowest Δ $5/2^-$ level differs by 13%, but shows convergence toward the experimental value. A larger basis may remedy these large differences.

The 6ω results may be summarized as follows. For the 26 experimental N and Δ levels which are described by eigenvectors with no less than three oscillator shells in their bases, the average, relative difference between theory and experiment is 5.4%. These bases have at least one state with E = 2 for $\pi = +1$ or one with E = 1 for $\pi = -1$. For the 34 levels which are described by eigenvectors with at least two oscillator shells in their bases, the average difference is 5.7%. These bases have at least one state with E = 4 for $\pi = +1$ or one with E = 3 for $\pi = -1$. For all of the 38 experimental levels, the average difference is 6.4%. This pattern suggests that a larger basis, 8ω or 10ω for example, could further reduce the overall difference between theory and experiment. However, because they include higher oscillator shells, these larger bases may admit resonant states in which the quark kinetic energies are too large to justify the exclusion of higher order, relativistic terms in H. In this case, the model would have to be modified to include these higher order terms. Note that the experimental masses in Table 4.3 are the only ones, as of April 1980 (4), for which J^π values may be assigned with some confidence. Thus, a 6ω basis encompasses all known N and Δ J^π levels.

The eigenvectors corresponding to the 6ω states of Table 4.3 are listed in Table 10.3. The notation used is $[M_R> = \sum_I e_I [I>$, where M_R is

the mass of the resonance, [1> is the basis state of Table 10.2, and e_1 is the coefficient of [1> in the eigenvector. It is hard to obtain any physical properties of the eigenvectors in Table 10.3. Thus, we will extract certain e_1 in the sums over basis functions to give properties of the eigenvectors which are easy to comprehend. We do this by classifying each basis function as to its L value, S value, and space S_3 irrep, j. For example, the state [1> in the N $1/2^+$ basis has (L,S,j) = (0,1/2,[3]). Then, we sum the squares of all the coefficients in an eigenvector which correspond to basis states with equal (L,S,j). For example, in the N $1/2^+$ basis, the states labeled [1> through [7> all have (L,S,j) = (0,1/2,[3]). Summing the squares of the coefficients of basis states [1> through [7> in the N(954) $1/2^+$ eigenvector give 0.99. Thus, this eigenvector is said to be 99% (L,S,j) = (0,1/2,[3]). Table 10.4 lists the major percentages of basis functions in each eigenvector of Table 10.3 by using this classification scheme.

In Table 4.4, we list the expectation values, as defined by (4.62), of the various terms in H for the 6ω basis. KE1 and KE2 are the expectation values of the p^2 and p^4 terms in the kinetic energy operator of (4.5), respectively. LC is that of V_{ij}^{LC} in (4.6) and GL1 that of the nonrelativistic, 1/r term in V_{ij}^{OGE} of (4.7). CNT, TNF, DAR, and SO are those of the Fermi-contact, tensor-force, Darwin, and spin-orbit terms, respectively, of (4.8)-(4.11). GL2 is that of the first order, relativistic correction in V_{ij}^{OGE}, defined as

$$GL2 = CNT + TNF + DAR + SO \quad .$$

Table 4.4. Expectation values, in units of MeV, of various terms in H and estimates of v^2 using the eigenvectors of Table 10.3. Experimental masses and widths are included for comparison

	v^2	KE1	KE2	LC	QL1	CNT	TNF	DAR	SO	QL2	Mass	Exp(width)
N 1/2+	0.41	414	-130	833	-908	36	-1	-249	0	-215	954	939(0)
	0.43	456	-148	1283	-551	26	-58	-194	-190	-417	1583	1470(200)
	0.42	450	-151	1263	-595	28	17	-225	-59	-238	1689	1710(120)
	0.43	448	-137	1294	-522	5	10	-208	-28	-221	1822	
N 1/2-	0.40	391	-113	1111	-652	64	-57	-180	-169	-341	1357	1535(150)
	0.40	386	-109	1110	-655	28	19	-179	-74	-206	1486	1650(150)
	0.40	391	-115	1565	-508	-6	0	-176	-31	-214	2080	2100(?)
	0.40	392	-114	1592	-437	10	-14	-161	-81	-245	2149	
N 3/2+	0.43	452	-146	1278	-561	5	-28	-182	-103	-308	1676	1540(?)
	0.43	451	-147	1274	-566	12	-3	-164	-92	-247	1725	1810(200)
	0.42	449	-151	1261	-590	75	19	-223	-44	-173	1757	
	0.43	447	-144	1273	-561	51	12	-172	-43	-153	1823	
N 3/2-	0.40	384	-108	1107	-654	47	24	-177	-53	-159	1531	1520(125)
	0.40	382	-107	1108	-654	46	3	-176	15	-113	1577	1700(120)
	0.40	390	-115	1563	-515	2	0	-178	6	-171	2113	2040(?)
	0.41	391	-111	1595	-441	21	-17	-129	-86	-211	2183	
N 5/2+	0.43	446	-144	1270	-579	-30	1	-158	29	-159	1794	1688(130)
	0.43	447	-141	1276	-553	44	20	-169	-39	-143	1845	2000(?)
	0.43	443	-140	1275	-556	46	-2	-166	46	-76	1907	
	0.42	440	-143	1719	-457	6	0	-168	-5	-166	2353	
N 5/2-	0.40	376	-104	1105	-653	85	-8	-173	96	0	1685	1670(155)
	0.41	390	-109	1595	-447	-4	-15	-107	-62	-188	2202	2100(?)
	0.41	389	-109	1595	-448	7	-9	-108	-50	-159	2228	
	0.40	389	-110	1593	-448	27	2	-114	-32	-117	2267	
N 7/2+	0.43	439	-135	1277	-551	44	-9	-169	124	-9	1982	1990(250)
	0.42	442	-143	1729	-426	-15	-6	-96	-43	-159	2403	
	0.42	441	-141	1737	-416	16	-12	-110	-48	-153	2428	
	0.42	438	-143	1717	-456	56	-2	-159	29	-76	2442	
N 7/2-	0.40	388	-110	1586	-464	-8	0	-97	21	-85	2275	2190(250)
	0.40	388	-111	1583	-462	37	8	-97	-11	-63	2296	
	0.41	387	-103	1606	-438	18	0	-103	18	-66	2346	
	0.41	388	-104	1612	-415	5	2	-117	36	-75	2367	
N 9/2+	0.42	440	-143	1725	-434	-18	1	-90	13	-95	2452	2220(300)
	0.42	441	-142	1729	-424	30	6	-93	-7	-64	2500	
	0.42	439	-140	1730	-425	29	0	-92	24	-38	2527	
	0.43	440	-129	1767	-379	7	9	-106	5	-85	2574	
N 9/2-	0.40	386	-111	1581	-466	40	-3	-96	58	0	2351	2200(250)
	0.38	344	-92	2148	-347	-9	-1	-49	-22	-81	2932	2800(?)
	0.38	343	-92	2150	-344	17	-3	-53	-28	-67	2951	
	0.38	339	-89	2157	-350	20	-3	-93	29	-47	2971	
N 11/2+	0.42	438	-139	1731	-425	32	-3	-91	61	-2	2563	
	0.43	439	-129	1767	-373	3	-5	-109	97	-14	2651	
	0.40	387	-117	2271	-336	-11	0	-46	-17	-75	3091	
	0.39	379	-112	2273	-336	33	-1	-95	29	-35	3109	

Table 4.4. Continued

	v^2	KE1	KE2	LC	QL1	CNT	TNF	DAR	SO	QL2	Mass	Exp(width)
N 11/2-	0.38	344	-93	2146	-349	-6	0	-49	11	-43	2965	2600(400)
	0.38	344	-93	2145	-350	20	3	-49	1	-26	2980	
	0.38	344	-90	2158	-343	13	0	-47	9	-26	3004	
	0.38	344	-89	2180	-310	1	1	-63	19	-42	3043	
N 13/2+	0.40	387	-117	2270	-336	-10	0	-46	8	-48	3116	2700(?)
	0.40	387	-116	2274	-334	19	2	-46	0	-25	3147	
	0.40	387	-116	2276	-334	18	0	-45	12	-14	3159	
	0.40	387	-110	2317	-293	1	5	-59	3	-49	3211	
N 13/2-	0.38	344	-93	2144	-351	21	-1	-49	33	4	3008	
	0.39	344	-82	2216	-291	1	-3	-61	57	-6	3141	
N 15/2+	0.40	387	-116	2276	-334	20	-1	-45	32	5	3179	
	0.40	387	-111	2317	-291	1	-3	-61	57	-6	3256	
Δ 1/2+	0.42	450	-151	1260	-589	81	-18	-161	-99	-197	1733	1550(?)
	0.42	448	-153	1254	-605	98	4	-230	-12	-141	1763	1910(220)
	0.42	441	-151	1707	-458	51	-2	-205	-9	-166	2335	
	0.43	440	-135	1758	-388	9	-6	-153	-55	-205	2430	
Δ 1/2-	0.40	383	-107	1105	-654	85	-1	-176	-23	-115	1571	1650(140)
	0.40	390	-115	1560	-517	91	4	-180	-41	-126	2151	1900(?)
	0.40	387	-109	1596	-446	27	0	-148	-27	-148	2241	
	0.31	246	-56	1771	-396	20	0	-87	-7	-74	2452	
Δ 3/2+	0.40	399	-122	826	-905	255	-6	-239	0	10	1168	1232(115)
	0.42	448	-149	1262	-584	75	-3	-153	-77	-157	1781	1690(250)
	0.43	447	-142	1275	-553	44	5	-169	-26	-146	1842	1960(?)
	0.32	281	-79	1405	-519	60	1	-124	-1	-63	1985	
Δ 3/2-	0.40	381	-107	1105	-653	84	-2	-175	11	-82	1605	1670(200)
	0.40	390	-115	1561	-514	89	-6	-178	-21	-117	2166	
	0.41	390	-105	1605	-436	27	-10	-105	-94	-181	2233	
	0.40	384	-106	1600	-444	27	4	-143	-4	-115	2278	
Δ 5/2+	0.42	445	-147	1262	-582	75	10	-151	-18	-84	1854	1890(250)
	0.43	445	-139	1275	-553	44	-2	-169	21	-106	1883	
	0.42	439	-143	1714	-461	61	-2	-174	4	-110	2400	
	0.42	442	-147	1718	-437	39	-4	-90	-63	-119	2419	
Δ 5/2-	0.40	388	-114	1561	-518	92	0	-179	22	-66	2212	1960(200)
	0.41	389	-110	1589	-456	36	-3	-99	-46	-111	2262	
	0.41	389	-106	1597	-446	32	5	-100	-32	-95	2299	
	0.41	389	-100	1625	-400	0	0	-130	24	-105	2368	
Δ 7/2+	0.42	440	-144	1261	-586	79	-5	-148	72	-2	1929	1950(240)
	0.42	442	-146	1721	-434	37	0	-89	-41	-93	2450	
	0.42	441	-142	1733	-416	24	0	-97	-9	-83	2494	
	0.43	441	-131	1763	-385	12	0	-104	-30	-122	2527	
Δ 7/2-	0.40	388	-111	1583	-462	39	2	-96	5	-51	2307	
	0.41	387	-103	1604	-440	30	4	-101	10	-56	2352	
	0.37	338	-90	2136	-383	43	0	-99	-1	-57	2905	
	0.38	340	-89	2153	-353	21	0	-103	6	-77	2935	

Table 4.4. Continued

	v^2	KE1	KE2	LC	QL1	CNT	TNF	DAR	BO	QL2	Mass	Exp(width)
Δ 9/2+	0.42	441	-145	1720	-436	38	4	-88	-1	-48	2492	2300(?)
	0.42	440	-141	1731	-420	27	0	-95	6	-62	2508	
	0.43	440	-129	1766	-379	8	-1	-106	32	-66	2591	
	0.39	379	-112	2273	-355	33	-1	-95	4	-59	3086	
Δ 9/2-	0.41	385	-101	1607	-438	30	-4	-101	71	-3	2409	2500(?)
	0.37	338	-91	2135	-382	42	0	-98	23	-33	2928	
	0.38	344	-93	2147	-348	19	-1	-50	-20	-51	2959	
	0.38	344	-90	2158	-343	20	2	-47	-15	-41	2988	
Δ 11/2+	0.42	439	-145	1719	-438	39	-2	-88	50	0	2535	2420(300)
	0.39	387	-118	2269	-336	20	0	-47	-21	-47	3115	
	0.40	387	-116	2276	-333	19	0	-46	-4	-32	3143	
	0.39	375	-107	2305	-330	22	-2	-90	39	-31	3172	
Δ 11/2-	0.38	344	-93	2145	-350	21	1	-49	6	-21	2984	
	0.38	344	-90	2159	-343	20	1	-47	8	-18	3012	
	0.39	344	-82	2216	-291	1	0	-61	17	-43	3105	
	0.39	344	-80	2228	-283	0	-1	-69	45	-25	3144	
Δ 13/2+	0.39	387	-118	2268	-337	20	2	-47	2	-22	3138	
	0.40	387	-116	2277	-331	18	0	-47	3	-26	3151	
	0.40	387	-111	2315	-294	3	0	-58	17	-39	3218	
	0.41	387	-101	2360	-269	1	3	-59	10	-46	3291	
Δ 13/2-	0.38	344	-89	2159	-343	20	-1	-47	33	5	3037	2750(?)
Δ 15/2+	0.39	387	-118	2267	-338	21	-1	-46	30	4	3163	2950(?)
	0.41	387	-101	2360	-269	1	-2	-60	53	-8	3329	

Thus, the resonance mass given in the next to last column is given by the equation

$$\text{Mass} = \text{KE1} + \text{KE2} + \text{LC} + \text{GL1} + \text{GL2} + 3m \quad ,$$

where $3m = 960$ MeV. The kinetic energy of one quark is

$$E_k = 1/3(\text{KE1} + \text{KE2})$$

and v^2 is then found by substituting this in (4.73). The experimental masses and widths are listed for comparison. In general, as the quark velocities increase, GL2 increases. This is expected since GL2 is a relativistic correction to V_{ij}^{OGE} and should become larger in magnitude as v^2 increases. Also, as J increases, LC increases and GL1 decreases. This indicates that the resonances with higher J values are larger particles (the quarks occupy a larger region of space) because as the average distance between quarks increases, $2/3k|\underset{\sim}{r}_i - \underset{\sim}{r}_j|$ increases and $-2/3\alpha_s|\underset{\sim}{r}_i - \underset{\sim}{r}_j|^{-1}$ decreases. The quark velocities, to a small extent, decrease as J increases, indicating that quarks move more slowly in larger resonances.

Let us consider a typical KE1 value in Table 4.4, that of N(954) $1/2^+$. Since this is the expectation value of $\sum_i \frac{p_i^2}{2m}$, then the value of p_i^2/m^2 may be taken as

$$\frac{2}{m}\left[\frac{1}{3}\,\text{KE1}\right] = 0.8625 \quad .$$

By expanding E_k to p^6 (see (2.12)), we find that this term in E_k has the value

$$3m\left[\frac{1}{16}\left(\frac{p_i^2}{m^2}\right)^3\right] = 38 \text{ MeV}$$

for N(954) $1/2^+$. This probably overestimates what the neglected p^6 term would contribute to H since the p^2 and p^4 terms approximate the fully relativistic kinetic energy. Hence, let us assume that the p^6 term would contribute about 25 MeV, on average, to the resonant masses. The effect of including the next order term in the one-gluon exchange is harder to estimate. Generally, KE1 and GL1 are about equal for most resonances in Table 4.4, as are KE2 and GL2. Hence, let us also assume that the next order, one-gluon exchange term would also contribute about 25 MeV, on average. From these arguments concerning the $1/c^4$ terms, we infer that the model is accurate to roughly 50 MeV if the sum of the contributions of all terms of higher order than $1/c^4$ is negligible. The accuracy observed in Table 4.3 is about 90 MeV for the 26 experimental levels described by eigenvectors with at least three oscillator shells in their bases. Since we feel that 50 MeV accuracy is sufficient, this shows that the quark velocities, as assumed in Chapter II, are of the magnitude in which only the first order, relativistic corrections to E_k and V_{ij}^{OGE} are important. Hence, the results are consistent with this underlying assumption.

Some miscellaneous results have also been calculated. For the n, we find that

$$r_{Q\text{-}n} = -0.074 \text{ F}$$

and experimentally,

$$r_{Q-n} = -0.336\ F$$

Hence, we are well below the accepted magnitude of r_{Q-n}. However, Isgur, Karl, and Koniuk (22) have shown that only a 7% mixing of (L,S,j) = (0,1/2,[2,1]) basis functions into the eigenvector can give the correct ratio of $r_{Q-n}/r_{Q-p} = -0.38$. We have this ratio as -0.09 because only 1% of these basis states are present in the N(954) $1/2^+$ eigenvector. Physically, the [2,1] space irrep is one in which the two equally charged quarks in the p or n repel and leave the third quark in the center. The [3] irrep is one in which the three quarks are separated by equal distances. Hence, by examining (4.57) for the n, we see that centering the u (Q = 2/3) between the two d (Q = -1/3) in the [2,1] irrep gives a negative contribution to the mean square charge radius, while the [3] irrep gives none. Note that centering the d between the two u will cause the p rms charge radius to be more positive. Hence, the inclusion of a larger percentage of (0,1/2,[2,1]) basis functions in the N(954) $1/2^+$ would bring r_{Q-p} and r_{Q-n} into closer agreement with experiment. The only term in H which causes the mixing of (0,1/2,[2,1]) basis functions with (0,1/2,[3]) is the Fermi-contact term. This implies that the strength of the Fermi-contact term is too small relative to the other one-gluon exchange terms. However, in this model the strength of this term, as compared to the 1/r, Darwin, spin-orbit, and tensor-force terms, is not adjustable. Isgur, Karl, and Koniuk avoided this by using only the Fermi-contact and tensor-force terms as their one-gluon exchange.

For the $\Delta(1168)\ 3/2^+$,

$$\mu_{\Delta^+} = \frac{1}{\sqrt{2}}\mu_{\Delta^{++}} = -\mu_{\Delta^-} = 2.925 \quad ,$$

$$\mu_{\Delta^0} = 0 \quad ,$$

$$r_{Q\text{-}\Delta^+} = \frac{1}{\sqrt{2}} r_{Q\text{-}\Delta^{++}} = -r_{Q\text{-}\Delta^-} = 0.779 \text{ F} \quad ,$$

and

$$r_{Q\text{-}\Delta^0} = 0 \quad .$$

The rms radius is defined as the square root of the expectation value of the operator

$$r_1^2 + r_2^2 + r_3^2 = \lambda^2 + \rho^2 \quad .$$

This is easily found by using (3.10), (3.11), the KE1 values of Table 4.4, and the eigenvectors of Table 10.3. The rms radius is 1.38 F for $N(954)\ 1/2^+$ and 1.34 F for $\Delta(1168)\ 3/2^+$. The levels with high J values are larger particles, as was suggested earlier. For example, the rms radius of $N(3116)\ 13/2^+$ is 3.56 F. This gives a better feeling for the spatial extent of these particles than does the rms charge radius, a measure of the distribution of charge.

The Ω resonances, containing three, identical s quarks, are similar to the Δ resonances in that they have symmetric isospin states. Hence, the Ω J^π basis functions are the same as those for Δ J^π, but with the isospin function replaced by one with three isospin 0 quarks. Then by

diagonalizing H, with m replaced by m_s, in a 6ω basis using the same values of α_s, β, and k in Table 4.2 and varying m_s to fit the Ω(1672) $3/2^+$ level exactly, we find m_s = 510 MeV. Note that this implies

$$\frac{m}{m_s} = 0.627$$

which is quite similar to DeRujula, Georgi, and Glashow's value (6) of 0.622. This then predicts that for the Ω(1672) $3/2^+$,

$$\mu_\Omega = -1.839$$

and

$$r_{Q-\Omega} = -0.773 \text{ F} \quad ,$$

where $T(\tau_{12},M_\tau;\ \tau',\tau)$ in (4.67)-(4.68) has been replaced by -1/3. Also, its rms radius is predicted to be 1.33 F. This is about equal to those of N(954) $1/2^+$ and Δ(1168) $3/2^+$. Since the s quarks are more massive than the u and d, the quark velocities in this particle are reduced to $v^2 = 0.23$.

V. PHOTON DECAYS

In Chapter IV, we compared the predicted mass eigenvalues of H in a 6ω basis with experiment and found decent agreement. However, agreement in this type of comparison may only show that H correctly reproduces the internal symmetries of the system. It does not guarantee that H also gives the physical processes correctly. To test this, calculations of physical quantities, which depend on the predicted eigenvectors, must be made. A model with the correct symmetries, but the wrong physical processes, will do poorly when comparing such quantities to the experimental results.

We have already given the N(954) $1/2^+$ eigenvector this type of test in that μ_p, μ_n, r_{Q-p}, and r_{Q-n} have been calculated. Good agreement with experiment, except for r_{Q-n}, was found. Thus, we assume that the eigenvector describing the N ground state (the p or n) is essentially correct, although somewhat deficient in the amount of (L,S,j) = (0,1/2,[2,1]) basis functions it contains. However, the rest of the eigenvectors need to be tested. Since the N(954) $1/2^+$ eigenvector is assumed to be correct, calculating decay amplitudes for the transition from an N or Δ resonance to N(954) $1/2^+$ via photon emission will provide such a test. The only other calculation of this type has been performed by Koniuk and Isgur (12) with reasonably good results. Our calculation will basically follow their procedure. They also calculated amplitudes for meson decay; however, the operator for meson decay involves some free parameters while that for photon decay does not. Hence, photon decay is a better test of the eigenvectors.

We begin by specifying a coordinate system which will simplify the calculations. It is chosen so that the direction of photon emission is along $\hat{z}$. Also, the inertial frame is chosen so that prior to decay, the momentum of the center of mass of the resonance, $\underset{\sim}{P}_R$, is zero. Thus, the photon momentum in pseudo-spherical form is

$$\underset{\sim}{K} = K\hat{e}_0^* \quad . \tag{5.1}$$

By conservation of momentum, the recoil momentum of the N(954) $1/2^+$ is

$$\underset{\sim}{P}_N = -K\hat{e}_0^* \quad . \tag{5.2}$$

By (1.3), the total energy of the photon is K and the total energy of the resonance prior to decay is its rest mass, denoted as M_R. Then by conservation of energy, we can show that

$$K = M_N \left(\frac{X^2 - 1}{2X} \right) \quad , \tag{5.3}$$

where $X = M_R/M_N$ and $M_N = 954$ is the mass of the p or n in this model. Without loss of generality, we then restrict ourselves to examining photons with positive helicity, or right-handed polarization. Hence, the photon polarization vector is

$$\underset{\sim}{\varepsilon} = \hat{e}_1^* \tag{5.4}$$

since positive helicity implies $\hat{e}_1 \cdot \underset{\sim}{\varepsilon} = 1$.

The operator for the nonrelativistic quark-photon decay process, as given by Koniuk and Isgur, is

$$\hat{O}_\gamma = -\frac{ie}{m}\sqrt{\frac{2\pi}{K}}\sum_{i=1}^{3} Q_i\, e^{-i\underset{\sim}{K}\cdot\underset{\sim}{r}_i}\,(\underset{\sim}{s}_i\cdot(\underset{\sim}{K}\times\underset{\sim}{\varepsilon}) + i\underset{\sim}{p}_i\cdot\underset{\sim}{\varepsilon}) \quad , \tag{5.5}$$

where e is the p charge, m is the quark mass, and $\underset{\sim}{r}_i$, $\underset{\sim}{p}_i$, and $\underset{\sim}{s}_i$ are the position, momentum, and spin of the ith quark, respectively. By applying the symmetry property used in deriving (4.3), substituting the Jacobi coordinates of Table 3.4 for $\underset{\sim}{r}_3$ and $\underset{\sim}{p}_3$, and substituting (4.59), we may rewrite this operator as

$$\hat{O}_\gamma = \frac{e}{m}\sqrt{\frac{2\pi}{K}}\,\delta^3(\underset{\sim}{K}+\underset{\sim}{P}_N)(3\tau_{3,0}+1/2)\, e^{i\sqrt{2/3}\,\underset{\sim}{K}\cdot\underset{\sim}{\lambda}}\,(K\underset{\sim}{s}_3 - \sqrt{2/3}\,\underset{\sim}{P}_\lambda) \quad , \tag{5.6}$$

where

$$\underset{\sim}{K}\times\underset{\sim}{\varepsilon} = iK\hat{\varepsilon}_1^* = iK\underset{\sim}{\varepsilon}$$

by (7.25) and (5.4). The factor $\delta^3(\underset{\sim}{K}+\underset{\sim}{P}_N)$ is the result of integrating over the center of mass variable and reflects the momentum conservation result, $\underset{\sim}{K}+\underset{\sim}{P}_N = 0$. This factor will be omitted in the following. By expanding the exponential in (5.6) in spherical harmonics (23) and using (7.27) to put the result in coupled tensor form, we find the form of the operator to be

$$\hat{O}_\gamma = \frac{4\pi e}{m}\sqrt{\frac{6\pi}{K}}\,(3\tau_{3,0}+1/2)\sum_{\ell=0}^{\infty} i^\ell(-1)^{\ell+1}\, j_\ell(\sqrt{2/3}\,K\lambda)\,\hat{\ell}$$
$$\times\,[[Y_\ell(\hat{\lambda})\times Y_\ell(\hat{K})]_0\times[\underset{\sim}{u}\times\underset{\sim}{\varepsilon}]_0]_{0,0} \quad , \tag{5.7}$$

where

$$\underset{\sim}{u} = K\underset{\sim}{s}_3 - \sqrt{2/3}\,\underset{\sim}{P}_\lambda \tag{5.8}$$

and j_ℓ is a spherical Bessel function (20). We then recouple the tensors in (5.7) by using (7.19). This gives the form of the operator as

$$\hat{O}_\gamma = \frac{4\pi e}{m}\sqrt{\frac{2\pi}{K}}\,(3\tau_{3,0}+1/2)\sum_{\ell\ell_1} i^\ell(-1)^{1+\ell+\ell_1}\,[j_\ell(\sqrt{2/3}\,K\lambda)Y_\ell(\hat{\lambda})\times \underset{\sim}{u}]_{\ell_1}$$

$$\cdot\,[Y_\ell(\hat{K})\times \underset{\sim}{\varepsilon}]_{\ell_1} \quad . \tag{5.9}$$

Since $\underset{\sim}{K}$ is in the direction of $\hat{z}$,

$$Y_{\ell m}(\hat{K}) = Y_{\ell m}(0,\phi) = \frac{\hat{\ell}}{\sqrt{4\pi}}\,\delta_{m,0} \quad .$$

By using this with (5.1), (7.7), and (7.27), we find

$$[j_\ell(\sqrt{2/3})Y_\ell(\hat{\lambda})\times \underset{\sim}{u}]_{\ell_1}\cdot[Y_\ell(\hat{K})\times \underset{\sim}{\varepsilon}]_{\ell_1} = -\frac{\hat{\ell}}{\sqrt{4\pi}}\,C(\ell 1\ell_1;\,011)$$

$$\times\,[j_\ell(\sqrt{2/3}\,K\lambda)Y_\ell(\hat{\lambda})\times \underset{\sim}{u}]_{\ell_1,-1}\,.$$

Substituting this into (5.9), we find the form of the operator to be

$$\hat{O}_\gamma = \frac{e}{m}\sqrt{\frac{8\pi^2}{K}}\,(3\tau_{3,0}+1/2)\sum_{\ell\ell_1} i^\ell(-1)^{\ell+\ell_1}\,\hat{\ell}C(\ell 1\ell_1;\,011)$$

$$\times\,[j_\ell(\sqrt{2/3}\,K\lambda)Y_\ell(\hat{\lambda})\times \underset{\sim}{u}]_{\ell_1,-1} \quad . \tag{5.10}$$

In nuclear physics, where photon energies are roughly 100 times smaller than we will have here, the long wavelength approximation is applied to the operator. To use this approximation, we must have the product of the photon energy and the radius of the emitting particle be much less than one. This then simplifies the Bessel functions since

$$j_\ell(ar) \propto (ar)^\ell \qquad \text{for} \qquad ar << 1 \quad .$$

However, the long wavelength approximation is not valid here. For a typical photon energy of 400 MeV and for $\lambda = 1.4$ F, roughly the N(954) $1/2^+$ rms radius, we find that the argument of the Bessel function has the maximum value $\sqrt{2/3}\ K\lambda = 2.32$. Note that this is the nonrelativistic quark-photon operator. To be entirely consistent with H, the next order, $1/c^2$ term in $\hat{O}_\gamma$ should be used. We will not do this since we feel that 25-30% accuracy is sufficient because the experimental data on photon decay are not known very precisely.

Before determining the matrix elements of $\hat{O}_\gamma$, we will define the function $R_B(N'L',NL;\ \ell,a)$, in an analogy with $R(N'L',NL;\ m)$ of (4.13), to be

$$R_B(N'L',NL;\ \ell,a) = N_{N'L'}N_{NL} \int_0^\infty dr\ r^{2+L+L'}\ L_{N'}^{L'+1/2}(r^2)\ L_N^{L+1/2}(r^2)\ e^{-r^2} \times j_\ell(ar) \quad ,$$

where

$$N_{NL} = [2N!/\Gamma(N + L + 3/2)]^{1/2} \quad ,$$

$$\underset{\sim}{L'} = \underset{\sim}{L} + \underset{\sim}{\ell} \quad ,$$

and

$$L + \ell + L' \text{ is even.}$$

Expanding the Laguerre polynomials by using (9.4), integrating, and using the equation (12)

$$\int_0^\infty dx\, x^m e^{-x^2} j_n(ax) = \frac{\sqrt{\pi}}{4} \left(\frac{a}{2}\right)^n \frac{\Gamma((m+n+1)/2)}{\Gamma(n+3/2)} e^{-a^2/4}$$

$$\times {}_1F_1((n-m+2)/2;\ n+3/2;\ a^2/4)$$

and the relation between the hypergeometric series and the Laguerre polynomials for a positive integer D (24)

$${}_1F_1(-D;\ A+1;\ C) = L_D^A(C)/\binom{D+A}{D} \quad ,$$

we find the explicit form of the function to be

$$R_B(N'L',NL;\ \ell,a) = \sqrt{\pi}\, 2^{-\ell-1}\, a^\ell (G(N',L')G(N,L))^{1/2}\, e^{-a^2/4}$$

$$\times \sum_{k'=0}^{N'} \sum_{k=0}^{N} (-1)^{k+k'} [(N'-k')!\,(N-k)!\,G(k',L')G(k,L)]^{-1}$$

$$\times D!\, L_D^{\ell+1/2}(a^2/4) \quad . \tag{5.11}$$

Here, we have defined

$$G(e,f) = e!\,\Gamma(e + f + 3/2)$$

and

$$D = 1/2(L + L' - \ell) + k + k' \quad .$$

The photon amplitude for the transition from a resonant state, $|J'M',\tau'M'_\tau\rangle$, to the N ground state, $|JM,\tau M_\tau\rangle$ for $J = \tau = 1/2$, is defined as the matrix element of $\hat{O}_\gamma$ between these eigenvectors,

$$A_{M'} = \langle J'M',\tau'M'_\tau|\hat{O}_\gamma|1/2,M,1/2,M_\tau\rangle \quad . \tag{5.12}$$

By applying (7.30), we see that the conservation of angular momentum demands $M' - M = -1$ because $\hat{0}_\gamma$ is the sum over the -1 components of tensors of rank ℓ_1. Since $J = 1/2$, M may be 1/2 or -1/2. If $J' = 3/2$, then M' may be 3/2 or 1/2. If $J' = 1/2$, then $M' = 1/2$ only. Since the photon is neutral, $M'_\tau = M_\tau$. For a transition to the p, $M_\tau = 1/2$, and for one to the n, $M_\tau = -1/2$. If the initial particle is a N resonance, $\tau' = 1/2$, then two different amplitudes occur for a given M'_τ. The first is labeled $A^p_{M'}$ and the second $A^n_{M'}$. The former is the amplitude for $M_\tau = 1/2$ and the latter for $M_\tau = -1/2$. If the initial particle is a Δ resonance, $\tau' = 3/2$, the $M_\tau = 1/2$ and $M_\tau = -1/2$ amplitudes are equal. In this case, we take $M_\tau = 1/2$ and label the amplitude as $A^\Delta_{M'}$. The matrix element of (5.12) is found by solving for the fundamental matrix elements of $\hat{0}_\gamma$ as given in the right hand side of (4.2),

$$\langle(\tau'_{12})\tau' M_\tau|\langle[L'S']J'M'|\hat{0}_\gamma|[LS]JM\rangle|(\tau_{12})\tau M_\tau\rangle \quad , \tag{5.13}$$

and then using (4.2), (4.62), and Table 10.3.

By using the definition of T in (4.66) and using (7.30) and (5.8), we find the matrix element in (5.13) to have the form

$$\frac{e}{m}\sqrt{\frac{8\pi^2}{K}}\, T(\tau_{12}, M_\tau;\ \tau', \tau)\delta_{\tau'_{12}\tau_{12}} \sum_{\ell\ell_1} i^\ell (-1)^{\ell+\ell_1} C(\ell 1 \ell_1;\ 011)$$

$$\times C(J\ell_1 J';\ M,-1,M')\left(K\langle[L'S']J'||[j_\ell(\sqrt{2/3}\,K\lambda)Y_\ell(\hat{\lambda}) \times s_3]_{\ell_1}||[LS]J\rangle\right.$$

$$\left. - \sqrt{2/3}\,\langle[L'S']J'||[j_\ell(\sqrt{2/3}\,K\lambda)Y_\ell(\hat{\lambda}) \times P_\lambda]_{\ell_1}||[LS]J\rangle\right) \quad . \tag{5.14}$$

The first reduced matrix element in (5.14) is found by using (7.13),

(7.32), and (7.33) to extract the reduced matrix elements depending on the $\underset{\sim}{\lambda}$ and $\underset{\sim}{s}_3$ variables. Then, (7.16) and (5.11) are used to solve the $\underset{\sim}{\lambda}$-space reduced matrix element and (7.31) is used to solve the $\underset{\sim}{s}_3$-space reduced matrix element. The result is

$$\langle [L'S']J' || [j_\ell(\sqrt{2/3}\, K\lambda) Y_\ell(\hat{\lambda}) \times \underset{\sim}{s}_3]_{\ell_1} || [LS]J \rangle$$

$$= \delta_{S'_{12}S_{12}} \delta_{L'_\rho L_\rho} \delta_{N'_\rho N_\rho} (-1)^{\ell+L_\rho+L_\lambda+L'+S_{12}+3/2+S} \sqrt{\frac{3}{8\pi}}$$

$$\times \hat{\ell}\hat{\ell}_1 \hat{L}_\lambda \hat{L}\hat{L}' \hat{S}\hat{S}' \hat{J}\, C(L_\lambda \ell L'_\lambda; 000) W(L_\lambda L'_\lambda LL'; \ell L_\rho) W(1/2,1/2,SS'; 1S_{12})$$

$$\times R_B(N'_\lambda L'_\lambda, N_\lambda L_\lambda; \ell, d) \begin{Bmatrix} L & S & J \\ \ell & 1 & \ell_1 \\ L' & S' & J' \end{Bmatrix} , \qquad (5.15)$$

where

$$d = \sqrt{\frac{2}{3}}\, \frac{K}{\beta} .$$

The second reduced matrix element in (5.14) is found by using (7.33) to extract the $\underset{\sim}{\lambda}$-space reduced matrix element and (7.35) to separate the $\underset{\sim}{P}_\lambda$ and $j_\ell Y_\ell$ operators. Then, (4.47) is used to solve the $\underset{\sim}{P}_\lambda$ matrix element and (7.16) and (5.11) are used to solve the $j_\ell Y_\ell$ matrix element. The result is

$$\langle [L'S']J' || [j_\ell(\sqrt{2/3}\, K\lambda) Y_\ell(\hat{\lambda}) \times \underset{\sim}{P}_\lambda]_{\ell_1} || [LS]J \rangle$$

$$= \delta_{S'S} \delta_{S'_{12}S_{12}} \delta_{L'_\rho L_\rho} \delta_{N'_\rho N_\rho} (-1)^{1+\ell+\ell_1+L_\rho+L_\lambda+L+L'+S+2J-J'} \frac{i}{\sqrt{4\pi}}$$

$$\times \hat{\ell}\hat{\ell}_1 \hat{L}\hat{L}' \hat{J}\, W(LL'JJ'; \ell_1 S) W(L_\lambda L'_\lambda LL'; \ell_1 L_\rho) \{\sqrt{(L_\lambda+1)(2L_\lambda+3)}$$

$$\times C(L_\lambda+1,\ell,L'_\lambda;\ 000)W(\ell 1L'_\lambda L_\lambda;\ \ell_1,L_\lambda+1)[\sqrt{N_\lambda+L_\lambda+3/2}$$

$$\times R_B(N'_\lambda L'_\lambda,N_\lambda,L_\lambda+1;\ \ell,d) + \sqrt{N_\lambda}\ R_B(N'_\lambda L'_\lambda,N_\lambda-1,L_\lambda+1;\ \ell,d)]$$

$$+ \sqrt{L_\lambda(2L_\lambda-1)}\ C(L_\lambda-1,\ell,L'_\lambda;\ 000)W(\ell 1L'_\lambda L_\lambda;\ \ell_1,L_\lambda-1)$$

$$\times [\sqrt{N_\lambda+L_\lambda+1/2}\ R_B(N'_\lambda L'_\lambda,N_\lambda,L_\lambda-1;\ \ell,d)$$

$$+ \sqrt{N_\lambda+1}\ R_B(N'_\lambda L'_\lambda,N_\lambda+1,L_\lambda-1;\ \ell,d)]\}\quad . \tag{5.16}$$

Substituting (5.15) and (5.16) into (5.14), we find the full matrix element of $\hat{0}_\gamma$ in (5.13).

The photon amplitudes, as computed from these matrix elements and the eigenvectors of Table 10.3, are given in Table 5.1. From Table 4.2, we have β = 165.8 MeV and m = 320.1 MeV and from Eq. (1.1), $e = \sqrt{1/137}$. The photon energies, K, are determined from (5.3). The amplitudes are given in units of 10^{-3} $GeV^{-1/2}$ and the experimental values and uncertainties, where known, are given just below the theoretical results. Any unknown experimental values or uncertainties are labeled with a question mark. The resonant eigenvectors have been multiplied by a phase factor given in the last column. This phase factor does not alter any physical quantities which may be calculated from the eigenvectors and is used because the signs of the experimental amplitudes are the result of conventions and cannot be measured. The amplitudes for resonances with $J' \geq 11/2$ are not listed because these are less than 3×10^{-3} $GeV^{-1/2}$ in magnitude.

Assuming about 25% accuracy in the results because $\hat{0}_\gamma$ is nonrelativistic, the theoretical values are in reasonable agreement with

Table 5.1. Photon decay amplitudes in units of 10^{-3} GeV$^{-1/2}$. Experimental results, where known, are given just below the theoretical results

	Mass (MeV)	$A^p_{1/2}$	$A^p_{3/2}$	$A^n_{1/2}$	$A^n_{3/2}$	Phase Factor
N 1/2+	1583	-7		6		-1
	1470	-77±10		35±22		
	1689	-40		20		1
	1710	2±40		9±50		
	1822	-13		9		1
N 1/2-	1357	78		-73		-i
	1535	60±19		-56±33		
	1486	120		-83		i
	1650	45±21		-22±17		
	2080	22		-8		-i
	2100	?		?		
	2149	-7		4		-i
N 3/2+	1676	67	-24	-39	11	1
	1540	?	?	?	?	
	1725	68	-24	-19	2	1
	1810	33±54	-39±43	9±40	-10±55	
	1757	23	-11	-2	23	1
	1823	-12	-1	-6	18	1
N 3/2-	1531	-23	71	-22	-101	i
	1520	-11±8	151±37	-75±15	-131±17	
	1577	-30	57	-5	-27	-i
	1700	-15±35	8±25	-4±45	12±30	
	2113	14	-11	-8	2	-i
	2040	?	?	?	?	
	2183	1	-1	1	2	-i
N 5/2+	1794	-12	61	17	-20	-1
	1688	-4±16	133±23	24±11	-20±20	
	1845	-2	11	21	24	1
	2000	?	?	?	?	
	1907	-1	11	-23	-20	1
	2353	-6	9	1	-7	1

Table 5.1. Continued

	Mass (MeV)	$A^p_{1/2}$	$A^p_{3/2}$	$A^n_{1/2}$	$A^n_{3/2}$	Phase Factor
N 5/2-	1685	3	4	-32	-43	-i
	1670	20±12	20±11	-30±26	-54±27	
	2202	-26	9	17	-6	-i
	2100	?	?	?	?	
	2228	22	-7	-9	5	-i
	2267	-7	3	5	-10	-i
N 7/2+	1982	-3	-4	-10	-12	1
	1990	40± ?	4± ?	-69± ?	-72± ?	
	2403	-14	7	7	-2	1
	2428	7	-3	-1	2	1
	2442	1	0	-3	-5	1
N 7/2-	2275	-7	17	-2	-14	i
	2190	-30± ?	180± ?	-85± ?	7± ?	
	2296	0	2	-4	-10	-i
	2346	-3	8	9	3	-i
	2367	2	1	3	-1	-i
N 9/2+	2452	-2	-10	0	4	1
	2220	?	?	?	?	
	2500	-1	2	4	3	1
	2527	-1	-4	4	5	1
	2574	0	-4	0	2	1
N 9/2-	2351	1	1	-5	-6	-i
	2200	?	?	?	?	
	2932	5	-2	-3	1	-i
	2800	?	?	?	?	
	2951	1	0	0	-1	-i
	2971	-1	0	-1	-2	-i

Table 5.1. Continued

	Mass (MeV)	$A^{\Delta}_{1/2}$	$A^{\Delta}_{3/2}$	Phase Factor
Δ 1/2+	1733	12		1
	1550	?		
	1763	-11		1
	1910	-14±23		
	2335	1		1
	2430	-1		1
Δ 1/2-	1571	55		i
	1650	39±45		
	2151	5		-i
	1900	?		
	2241	2		-i
	2452	-1		-i
Δ 3/2+	1168	-92	-159	1
	1232	-141±7	-259±10	
	1781	-25	15	-i
	1690	-8±20	-7±25	
	1842	26	-9	1
	1960	?	?	
	1985	3	13	1
Δ 3/2-	1605	86	89	-i
	1670	63±43	58±39	
	2166	3	13	-i
	2233	1	1	-i
	2278	6	2	-i
Δ 5/2+	1854	18	43	1
	1890	35±20	-7±60	
	1883	27	24	1
	2400	8	7	1
	2419	-1	0	1

Table 5.1. Continued

	Mass (MeV)	$A^{\Delta}_{1/2}$	$A^{\Delta}_{3/2}$	Phase Factor
Δ 5/2-	2212	7	12	i
	1960	62± ?	19± ?	
	2262	-9	9	-i
	2299	-3	3	-i
	2368	-2	-1	-i
Δ 7/2+	1929	-22	-27	-1
	1950	-71±15	-101±45	
	2450	2	-1	1
	2494	-8	5	1
	2527	-3	0	1
Δ 7/2-	2307	-18	-16	-i
	2352	-4	3	-i
	2905	-1	-4	-i
	2935	-2	-2	-i
Δ 9/2+	2492	-1	-2	1
	2300	?	?	
	2508	-5	-7	1
	2591	-4	-2	1
	3086	3	2	1
Δ 9/2-	2409	5	6	-i
	2500	?	?	
	2928	-3	-3	-i
	2959	-1	2	-i
	2988	0	0	-i

experiment, where uncertainties are known. The N(1583) $1/2^+$, Δ(1168) $3/2^+$, and Δ(1929) $7/2^+$ amplitudes are too small in magnitude and the N(1486) $1/2^-$ are too large, although the Δ(1168) $3/2^+$ amplitudes are not in serious disagreement. Koniuk and Isgur's results (12) are comparable to ours, but with slightly better agreement. Thus, the eigenvectors predicted to arise from H give decent agreement with experiment for the photon amplitudes. These results, taken with those of Table 4.3, imply that H correctly describes the basic quark interactions.

VI. CONCLUDING REMARKS

To summarize, H reproduces the entire, known N and Δ mass spectrum with about 6% accuracy using a 6ω basis. The p and n magnetic moments and the p rms charge radius are reproduced well, but the n rms charge radius is too small in magnitude. Also, the model is consistent with the assumption that only first order relativistic corrections are needed. Reasonable agreement between theory and experiment is also found for photon decay amplitudes. However, there are a few masses and photon amplitudes which differ significantly from experiment. These results show that for the most part, H correctly describes the basic interactions between quarks in nonstrange baryons, but further study is warranted.

At this point, we will briefly consider some ways in which the model could be extended in hope of improving agreement between theory and experiment. One would especially want to bring those theoretical results which differ significantly, such as r_{Q-n}, into better agreement.

The first, and most obvious, method is to diagonalize H in a larger basis after fixing the parameters through a chi-square minimization. As discussed in Chapter IV, the resulting kinetic energies must not be too large. Otherwise, higher order, relativistic terms would be needed. This procedure would probably bring the masses into better, overall agreement with experiment. However, the two lowest, N $1/2^-$ masses and r_{Q-n} would probably not improve.

The second method is to consider alternate forms of the confining potential. For example, a potential other than a linear one may better

reproduce the data. Also, relativistic corrections to the confining potential may be necessary. However, this is a hit-or-miss method. The fundamental origin and form of quark confinement is needed, at least in the sense that V_{ij}^{OGE} is given explicitly if we take only one-gluon exchange terms to order $1/c^2$, to perform a meaningful calculation.

Another method is the inclusion of the $1/c^4$ terms in E_k and V_{ij}^{OGE}. However, these V_{ij}^{OGE} terms would be quite complicated and the model would lose its simplicity. Note that the splittings between masses in Table 4.3 with the same J^{π} values are generally too small. For example, the splitting of the two lowest N $3/2^-$ masses is predicted to be 1577 - 1531 = 46 MeV and experimentally it is 1700 - 1520 = 180 MeV. These $1/c^4$ terms or relativistic corrections to the confining potential could increase these splittings and also bring r_{Q-n} and the N $1/2^-$ levels into better agreement with experiment.

By using the Hamiltonian of (2.13)-(2.15), generalized for unequal mass quarks, with the value of m_s as found in Chapter IV from the Ω^- mass and the four parameters of Table 4.2, the Λ and Σ resonant masses and magnetic moments could be calculated in a 6ω basis. There is a significant amount of experimental data available for comparison. Since we have the ratio m/m_s nearly equal to that of DeRujula, Georgi, and Glashow (6) and since our eigenvectors for the lowest N $1/2^+$ and Δ $3/2^+$ masses have nearly the same respective (L,S,j) compositions of 100% (0,1/2,[3]) and 100% (0,3/2,[3]) as they assumed, we would expect from their results that our model will also reproduce the Λ and Σ data

reasonably well. A lack of time and funds prevented us from performing these calculations.

It has been pointed out by D. L. Pursey that β, the inverse length, is not a parameter of the model since, unlike m, α_s, and k, it does not occur in H. He has suggested it be chosen such that the energy matrix element

$$\langle\Psi|H_o|\Psi\rangle$$

be minimal for fixed m, α_s, and k. H_o consists of the nonrelativistic terms in H and $|\Psi\rangle$ is the eigenvector of the N $1/2^+$ ground state. Then the $1/c^2$ terms in H would be considered a perturbation upon H_o. Note that $\langle\Psi|H|\Psi\rangle$ has no minimum value since the $1/c^2$ terms are negative and increase in magnitude as β^4 for E_k and β^3 for V_{ij}^{OGE} as compared to β^2 and β for the respective nonrelativistic terms. This calculation could be performed by fitting m, α_s, and k in a chi-square minimization with the constraint that $\langle\Psi|H_o|\Psi\rangle$ be minimal. Note that in our 6ω basis, given m, α_s, and k in Table 4.3, this matrix element is minimal for $\beta \approx 500$ MeV. This is a more complicated fitting procedure than we have employed. We used the more direct route of basically fixing β by constraining r_{Q-p} to be in close agreement with the experimental value. Also, if

$$H_o' = 3m + \frac{1}{2m}(P_\lambda^2 + P_\rho^2) + \frac{1}{2}m\omega^2(\lambda^2 + \rho^2)$$

is the two-particle, harmonic oscillator Hamiltonian and if

$$H' = H - H_o' \quad ,$$

we instead formally considered H' to be a perturbation upon H'_o because we have employed a harmonic oscillator basis.

If research shows that our semirelativistic quark model must be extended, the calculational procedures shown in this thesis will still be useful. Undoubtedly, the Hamiltonian will have to be diagonalized in a truncated, harmonic oscillator basis using overall antisymmetric functions in combined space, spin, isospin, and color variables for the N, Δ, and Ω resonances.

VII. APPENDIX A: ANGULAR MOMENTUM CONVENTIONS AND RESULTS

Conventions, notations, and some pertinent results from angular momentum theory are presented in this appendix. These are a summary of results to be found in the text by M. E. Rose (25) and in the unpublished nuclear theory notes by S. A. Williams. Presenting efficient methods for calculating matrix elements is the major goal of this appendix.

One begins with the fact that for an isolated system, such as two particles interacting with only internal forces, the total angular momentum of the system, $\underset{\sim}{J}$, is conserved. This is equivalent to writing that all components of $\underset{\sim}{J}$ commute with the Hamiltonian, H,

$$[H,J_i] = HJ_i - J_iH = 0 \quad \text{for} \quad i = 1,2,3 \quad . \tag{7.1}$$

The operator $\underset{\sim}{J}$ generates rotations of the system and then H is invariant under these rotations. Thus, transformation properties of wavefunctions and operators under rotations are of prime importance and this will provide a method of classification of wavefunctions and operators. The components of $\underset{\sim}{J}$ satisfy the commutation relations

$$[J_i,J_j] = iJ_k \quad \text{for} \quad (ijk) = (123) \quad , \tag{7.2}$$

cyclic. Angular momentum is expressed in units of $\hbar$ and $\hbar$ is set equal to one here. From this result, the following relation is derived for $\underset{\sim}{J}^2 = \underset{\sim}{J} \cdot \underset{\sim}{J}$,

$$[\underset{\sim}{J}^2,J_i] = 0 \quad \text{for} \quad i = 1,2,3 \quad . \tag{7.3}$$

Thus, H, $\underset{\sim}{J}^2$, and one component of $\underset{\sim}{J}$, usually chosen to be J_3, commute

with one another. This implies that eigenfunctions of H may be chosen, through a suitable unitary transformation if necessary, so that they are simultaneously eigenfunctions of $\underset{\sim}{J}^2$ and J_3. These eigenfunctions, ψ, will have corresponding labels J and M such that

$$\underset{\sim}{J}^2\psi_{JM} = J(J+1)\psi_{JM} \quad \text{and} \quad J_3\psi_{JM} = M\psi_{JM} \quad . \tag{7.4}$$

J labels the total angular momentum and M its projection on the z-axis. In general, J may take the values 0, 1/2, 1, 3/2, ... and for each J, M may take the 2J+1 values -J, -J+1, ..., J. The ψ_{JM} will be denoted as $|JM\rangle$ in the following. The labels J and M also classify the eigenfunctions according to their transformation properties under rotations. That is, if $R(\theta,\hat{n})$ is an operator for a rotation of angle θ about the axis $\hat{n}$, then

$$R(\theta,\hat{n})|JM\rangle = \sum_{M'} D^J_{M'M}(\theta,\hat{n})|JM'\rangle \quad . \tag{7.5}$$

Therefore, under rotations the 2J+1 eigenfunctions $|JM\rangle$ transform as linear combinations of each other. The $D^J_{M'M}$ are elements of a matrix D^J called a rotation matrix.

The next topic to be considered is the addition of two angular momenta. This can occur in the coupling of angular momenta of two different particles or that of two different degrees of freedom, such as orbital and spin angular momentum. In vector addition, one has

$$\underset{\sim}{j}' + \underset{\sim}{j}'' = \underset{\sim}{J} \tag{7.6}$$

with the constraint $[\underset{\sim}{j}', \underset{\sim}{j}''] = 0$. J may then take only the values

$|j'-j''|$, $|j'-j''|+1$, ..., $j'+j''$. For instance, $j' = 1$ and $j'' = 3/2$ can give rise to $J = 1/2$, $3/2$, or $5/2$. This constraint is denoted as $\Delta(j'j''J)$, since j', j'', and J must form the sides of a triangle. Applying a unitary transformation to the open product states $|j'm'\rangle|j''m'\rangle$, which are eigenfunctions of $\underset{\sim}{j}'^2$, $\underset{\sim}{j}''^2$, j_3', and j_3'', gives the coupled states

$$|[j' \times j'']JM\rangle = \sum_{m'm''} C(j'j''J;\ m'm''M)|j'm'\rangle|j''m'\rangle \tag{7.7}$$

which are eigenfunctions of $\underset{\sim}{J}^2$, $\underset{\sim}{j}'^2$, $\underset{\sim}{j}''^2$, and J_3. The coupled states may be denoted as $|(j'j'')JM\rangle$ also. These coupled states satisfy (7.4) for the operators $\underset{\sim}{J}^2$ and J_3 which implies that the coupled states have a definite value, J, of the total angular momentum and a definite projection value, M. The inverse transformation is

$$|j'm'\rangle|j''m'\rangle = \sum_{JM} C(j'j''J;\ m'm''M)|[j' \times j'']JM\rangle \quad . \tag{7.8}$$

The coefficients of the unitary transformation, $C(j'j''J;\ m'm''M)$, are Clebsch-Gordan coefficients and are zero unless $M = m' + m''$ and $\Delta(j'j''J)$ is satisfied. They also satisfy the following orthogonality relations,

$$\sum_{m'm''} C(j'j''J;\ m'm''M)C(j'j''J';\ m'm''M') = \delta_{JJ'}\delta_{MM'} \tag{7.9}$$

and

$$\sum_{JM} C(j'j''J;\ m'm''M)C(j'j''J;\ n'n''M) = \delta_{m'n'}\delta_{m''n''} \quad . \tag{7.10}$$

The Clebsch-Gordan coefficients satisfy many symmetry properties, two of which are

$$C(j'j''J;\ m'm''M) \;=\; (-1)^{j'+j''-J}\, C(j''j'J;\ m''m'M) \qquad (7.11)$$

and

$$C(j'j''J;\ m'm''M) \;=\; (-1)^{j'+j''-J}\, C(j'j''J;\ -m',-m'',-M) \quad . \qquad (7.12)$$

From (7.11), one may easily show

$$|[j' \times j'']JM\rangle \;=\; (-1)^{j'+j''-J}\,|[j'' \times j']JM\rangle \quad . \qquad (7.13)$$

Related coefficients are 3-j symbols, $\begin{pmatrix} j' & j'' & J \\ m' & m'' & M \end{pmatrix}$, and are given by the relation

$$C(j'j''J;\ m'm''M) \;=\; \hat{J}(-1)^{j'-j''+M} \begin{pmatrix} j' & j'' & J \\ m' & m'' & -M \end{pmatrix} \qquad (7.14)$$

where $\hat{J} = \sqrt{2J+1}$. By considering properties of the rotation matrices, one may show that for the spherical harmonics, $Y_{\ell m}(\theta,\phi)$,

$$Y_{\ell'm'}(\theta,\phi)Y_{\ell''m''}(\theta,\phi) \;=\; \sum_{LM} \frac{\hat{\ell}'\hat{\ell}''}{\sqrt{4\pi}\,\hat{L}}\, C(\ell'\ell''L;\ m'm''M)C(\ell'\ell''L;\ 000)Y_{LM}(\theta,\phi) \quad . \qquad (7.15)$$

$Y_{\ell m}(\theta,\phi)$ is an eigenfunction of $\underset{\sim}{L}^2$ and L_3 where $\underset{\sim}{L} = \underset{\sim}{r} \times \underset{\sim}{p}$. This result is similar to (7.8), but involves functions with the same variables. $C(\ell'\ell''L;\ 000)$ is called the parity coefficient because it is zero unless $\ell'+\ell''+L$ is an even integer. From (7.15) and the orthonormality of the Y_{LM},

$$\int d\Omega\; Y^*_{LM}\, Y_{\ell'm'}\, Y_{\ell''m''} \;=\; \frac{\hat{\ell}'\hat{\ell}''}{\sqrt{4\pi}\,\hat{L}}\, C(\ell'\ell''L;\ m'm''M)C(\ell'\ell''L;\ 000) \quad . \qquad (7.16)$$

This result is used frequently in calculating matrix elements of operators in space variables.

Two types of coefficients of a higher order than the Clebsch-Gordan coefficients are needed. The first are Racah coefficients which arise in the recoupling of three angular momenta. They are denoted by $W(j_1 j_2 J j_3;\ J'J'')$ and are defined as coefficients of a unitary transformation in the equation

$$|[J' \times j_3]JM\rangle = \sum_{J''} \hat{J}'\hat{J}'' W(j_1 j_2 J j_3;\ J'J'') |[j_1 \times J'']JM\rangle \qquad (7.17)$$

where $\underset{\sim}{J}' = \underset{\sim}{j}_1 + \underset{\sim}{j}_2$ and $\underset{\sim}{J}'' = \underset{\sim}{j}_2 + \underset{\sim}{j}_3$. W(abcd; ef) is zero unless Δ(abe), Δ(bdf), Δ(edc), and Δ(afc) are satisfied. The 6-j symbols, $\{ \begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix} \}$, are related by the formula

$$W(abcd;\ ef) = (-1)^{a+b+c+d} \{ \begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix} \} \quad . \qquad (7.18)$$

The second type of coefficients are 9-j symbols These arise in the recoupling of four angular momenta. They are denoted by

$$\begin{Bmatrix} J_1 & J_2 & J_{12} \\ J_3 & J_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix}$$

and are the ceofficients of a unitary transformation in the equation

$$|[J_{12} \times J_{34}]JM\rangle = \sum_{J_{13}J_{24}} \hat{J}_{12}\hat{J}_{34}\hat{J}_{13}\hat{J}_{24} \begin{Bmatrix} J_1 & J_2 & J_{12} \\ J_3 & J_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix} |[J_{13} \times J_{24}]JM\rangle \qquad (7.19)$$

where $\underset{\sim}{J}_{12} = \underset{\sim}{J}_1 + \underset{\sim}{J}_2$, $\underset{\sim}{J}_{34} = \underset{\sim}{J}_3 + \underset{\sim}{J}_4$, $\underset{\sim}{J}_{13} = \underset{\sim}{J}_1 + \underset{\sim}{J}_3$, and $\underset{\sim}{J}_{24} = \underset{\sim}{J}_2 + \underset{\sim}{J}_4$. These are zero unless $\Delta(J_1J_2J_{12})$, $\Delta(J_3J_4J_{34})$, $\Delta(J_1J_3J_{13})$, $\Delta(J_2J_4J_{24})$, $\Delta(J_{12}J_{34}J)$, and $\Delta(J_{13}J_{24}J)$ are satisfied.

The Racah and 9-j coefficients find much utility in computing matrix elements of operators between coupled wavefunctions. However, the concept of an irreducible tensor operator is needed first. An irreducible tensor of rank L and component M depending on the variable x, $T_{LM}(x)$, is an operator which transforms under rotations in the same manner as $|JM\rangle$ in (7.5). That is,

$$R(\theta,\hat{n})T_{LM}(x)R^{\dagger}(\theta,\hat{n}) = \sum_{M'} D^{L}_{M'M}(\theta,\hat{n})T_{LM'}(x) \quad . \quad (7.20)$$

Two tensors may be coupled in the same manner as wavefunctions in (7.7),

$$T_{LM}(x',x'') = \sum_{m'm''} C(\ell'\ell''L;\, m'm''M)T_{\ell'm'}(x')T_{\ell''m''}(x'') \quad . \quad (7.21)$$

This may be denoted by $T_{LM}(x',x'') = [T_{\ell'} \times T_{\ell''}]_{LM}$. Thus, T_{LM} transforms as a tensor of rank L under simultaneous rotations in the x' and x'' variables. A tensor which has rank L = 0 transforms as a scalar and a tensor of rank L = 1 tranforms as the components of a vector. It is useful to consider vectors in more detail along this line and cast them into what is called pseudo-spherical form in which their components are the components of a first rank tensor, $T_{1\nu}$. This form arises from considering the vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and its relationship to the spherical harmonics $Y_{1m}(\theta,\phi)$ for m = 1,0,-1. Writing the spherical harmonics in terms of Cartesian coordinates, we find that

$$Y_{11}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\ \frac{1}{r}\left[-\frac{1}{\sqrt{2}}(x + iy)\right] \quad ,$$

$$Y_{1,-1}(\theta,\phi) = -Y^{*}_{11}(\theta,\phi) \quad ,$$

and

$$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\ \frac{1}{r}[z] \quad .$$

$\underset{\sim}{r}$ may then be written as

$$\underset{\sim}{r} = r\sqrt{\frac{4\pi}{3}} \sum_{\nu} Y_{1\nu}\underset{\sim}{e}^{*}_{\nu} = \sum_{\nu} X_{\nu}\underset{\sim}{e}^{*}_{\nu} \tag{7.22}$$

where

$$X_{\nu} = \sqrt{\frac{4\pi}{3}}\ rY_{1\nu} \quad .$$

$$\underset{\sim}{e}_{1} = -\frac{1}{\sqrt{2}}[\underset{\sim}{i} + i\underset{\sim}{j}] \quad ,$$

$$\underset{\sim}{e}_{-1} = -\underset{\sim}{e}^{*}_{1} \quad ,$$

and

$$\underset{\sim}{e}_{0} = \underset{\sim}{k}$$

are the pseudo-spherical unit vectors. These unit vectors satisfy

$$\underset{\sim}{e}^{*}_{\nu} = (-1)^{\nu}\underset{\sim}{e}_{-\nu} \quad , \tag{7.23}$$

$$\underset{\sim}{e}_{\mu} \cdot \underset{\sim}{e}^{*}_{\nu} = \delta_{\mu\nu} \quad , \tag{7.24}$$

and

$$\underset{\sim}{e}_\mu \times \underset{\sim}{e}_\nu = i\sqrt{2}\, C(111;\ \mu,\nu,\mu+\nu)\underset{\sim}{e}_{\mu+\nu} \quad . \tag{7.25}$$

Correspondingly, any vector A may be written in this form, $\underset{\sim}{A} = \sum_\nu A_\nu \underset{\sim}{e}^*_\nu$ and by (7.24), $A_\nu = \underset{\sim}{A} \cdot \underset{\sim}{e}_\nu$. A vector scalar product is then $\underset{\sim}{A} \cdot \underset{\sim}{B} = \sum_\nu (-1)^\nu A_\nu B_{-\nu}$. This can be represented as two coupled tensors in the notation of (7.21) as

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = -\sqrt{3}\, [T_1(A) \times T_1(B)]_{0,0} \tag{7.26}$$

where $A_\nu = T_{1\nu}(A)$ and $B_\nu = T_{1\nu}(B)$. A general scalar product of two Lth rank tensors is given as

$$T_L(x) \cdot T_L(y) = (-1)^L \hat{L} [T_L(x) \times T_L(y)]_{0,0} \quad . \tag{7.27}$$

The Mth component of a vector cross product can be written as a coupled tensor product by using (7.25),

$$\begin{aligned} (\underset{\sim}{A} \times \underset{\sim}{B})_M &= -i\sqrt{2} \sum_{\mu\nu} A_\mu B_\nu C(111;\ \mu\nu M) \\ &= -i\sqrt{2}\, [T_1(A) \times T_1(B)]_{1,M} \quad . \end{aligned} \tag{7.28}$$

Another coupled tensor product often encountered is the tensor-force operator, S_{12},

$$S_{12} = (\underset{\sim}{s}_1 \cdot \underset{\sim}{r})(\underset{\sim}{s}_2 \cdot \underset{\sim}{r}) - \frac{1}{3} (\underset{\sim}{s}_1 \cdot \underset{\sim}{s}_2) \underset{\sim}{r}^2$$

where $\underset{\sim}{s}_k$ is the spin particle k and $\underset{\sim}{r} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2$. This may be written as two second rank tensors coupled to zero,

$$S_{12} = \sqrt{5}\, [[\underset{\sim}{r} \times \underset{\sim}{r}]_2 \times [\underset{\sim}{s}_1 \times \underset{\sim}{s}_2]_2]_{0,0}$$

or as a scalar product by using (7.27),

$$S_{12} = [\underset{\sim}{r} \times \underset{\sim}{r}]_2 \cdot [\underset{\sim}{s}_1 \times \underset{\sim}{s}_2]_2 \quad . \tag{7.29}$$

The components of the vectors $\underset{\sim}{s}_k$ and $\underset{\sim}{r}$ are first rank tensors in this notation.

A result of prime importance is the Wigner-Eckart theorem concerning matrix elements of irreducible tensors. It states that

$$\langle j'm'|T_{LM}|jm\rangle = C(jLj';\ mMm')\langle j'||T_L||j\rangle \quad . \tag{7.30}$$

This means that a matrix element, $\langle j'm'|T_{LM}|jm\rangle$, can be written as a geometrical factor, $C(jLj';\ mMm')$, containing the angular momentum relationships such as $\Delta(jLj')$, multipled by a reduced matrix element, $\langle j'||T_L||j\rangle$, containing the physical properties apart from angular momentum considerations. A simple example is

$$\langle j'm'|J_\nu|jm\rangle = C(j1j';\ m\nu m')\langle j'||J||j\rangle \quad .$$

Using (7.4),

$$\langle j'm'|J_0|jm\rangle = \delta_{j'j}m$$

and comparing it to the right hand side gives

$$\langle j'||J||j\rangle = \delta_{j'j}\sqrt{j(j+1)} \tag{7.31}$$

where

$$C(j1j';\ m0m') = \delta_{m'm}\frac{m}{\sqrt{j(j+1)}} \quad .$$

The last set of results to be presented arise from the problem of finding reduced matrix elements of coupled tensors between coupled

wavefunctions and involves both Racah and 9-j coefficients. If $T_{LM} = [T_{\ell_1}(x_1) \times T_{\ell_2}(x_2)]$ and if j_k is the label associated with the variable x_k in the coupled wavefunction $|(j_1 j_2)JM\rangle$, then what is the formula for writing $\langle (j_1' j_2')J' || T_L || (j_1 j_2)J\rangle$ as a product of the simpler reduced matrix elements $\langle j_1' || T_{\ell_1}(x_1) || j_1\rangle$ and $\langle j_2' || T_{\ell_2}(x_2) || j_2\rangle$? In general, the result is

$$\langle (j_1' j_2')J' || T_L || (j_1 j_2)J\rangle = \hat{L}\hat{J}\hat{j}_1'\hat{j}_2' \begin{Bmatrix} j_1 & j_2 & J \\ \ell_1 & \ell_2 & L \\ j_1' & j_2' & J' \end{Bmatrix} \langle j_1' || T_{\ell_1}(x_1) || j_1\rangle \times \langle j_2' || T_{\ell_2}(x_2) || j_2\rangle \quad . \tag{7.32}$$

Two specialized results follow from this. The first is for $\ell_2 = 0$ which implies that T_L is proportional to $T_{\ell_1}(x_1)$,

$$\langle (j_1' j_2')J' || T_{\ell_1}(x_1) || (j_1 j_2)J\rangle = \delta_{j_2 j_2'} \langle j_1' || T_{\ell_1}(x_1) || j_1\rangle \times \hat{j}_1'\hat{J}(-1)^{\ell_1 + j_1 + j_2 + J' + 2J + 2j_1'} \times W(j_1 j_1' JJ';\ \ell_1 J_2) \quad . \tag{7.33}$$

The second is for $L = 0$, a dot product of two tensors, which implies that $\ell_1 = \ell_2$. If $T = T_\ell(x_1) \cdot T_\ell(x_2)$, then

$$\langle (j_1' j_2')J' || T || (j_1 j_2)J\rangle = (-1)^{j_1' + j_2 - J} \hat{j}_1'\hat{j}_2' \delta_{JJ'} W(j_1 j_1' j_2 j_2';\ \ell J) \langle j_1' || T_\ell(x_1) || j_1\rangle \langle j_2' || T_\ell(x_2) || j_2\rangle \quad . \tag{7.34}$$

Another useful result arises from considering the reduced matrix element of a coupled tensor with variables in the same space. If $T_{LM}(x) = [T_{\ell_1}(x) \times T_{\ell_2}(x)]_{LM}$, then

$$\langle \alpha' j' || T_L || \alpha j \rangle = (-1)^{\ell_1+\ell_2+L+2j-2j'} \hat{L} \sum_{\text{all } \alpha'', j''} \hat{j}'' W(\ell_1 \ell_2 j' j;\ L j'') \times \langle \alpha' j' || T_{\ell_1} || \alpha'' j'' \rangle \langle \alpha'' j'' || T_{\ell_2} || \alpha j \rangle \quad . \qquad (7.35)$$

The labels α, α', and α'' may represent quantum numbers other than the angular momentum quantum numbers j and m in the wavefunctions $\psi_{\alpha jm}(x)$. For example, α could be the radial quantum number. These last four results are the most important for this thesis because they provide powerful methods for calculating matrix elements. They factor out all of the tensor and wavefunction coupling coefficients and leave the physical, bare, reduced matrix elements.

VIII. APPENDIX B: THE SYMMETRIC GROUPS S_2 AND S_3

In physics, a system of n identical particles is often encountered. The Hamiltonian describing this system will then be invariant under any permutation of the n particles. Suppose the system is one consisting of three particles without spin and of equal mass interacting with harmonic forces of equal spring constants

$$H = \frac{1}{2m}(\underset{\sim}{p}_1^2 + \underset{\sim}{p}_2^2 + \underset{\sim}{p}_3^2) + \frac{1}{2}K((\underset{\sim}{r}_1 - \underset{\sim}{r}_2)^2 + (\underset{\sim}{r}_2 - \underset{\sim}{r}_3)^2 + (\underset{\sim}{r}_3 - \underset{\sim}{r}_1)^2) \quad . \tag{8.1}$$

This Hamiltonian is invariant under any interchange of the three particle indices. This symmetry property of the Hamiltonian can then be used to classify wavefunctions of the system, $\psi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3)$, as to their transformation properties under interchange of particle indices. This concept is familiar from Bose-Einstein and Fermi-Dirac statistics. The wavefunction describing the system is symmetric (invariant) under the exchange of any two particles in Bose-Einstein statistics and is antisymmetric (changes sign) in Fermi-Dirac statistics. Projection operators, objects which project out functions with these transformation properties from an orthonormal set of functions, are needed. The goal of this appendix is to give these projection operators for systems of two and three identical particles. Much of the following is taken from Chapters 3 and 7 of the text by M. Hamermesh (26) and also from the unpublished group theory notes by D. L. Pursey and S. A. Williams.

To begin, the concept of a group is needed. A group G is a set of transformation operators (such as particle permutations), g, under a

binary "multiplication" law and satisfying the following rules for π_1, π_2, and π_3 in the set g.

(1) Multiplication is closed. The product $\pi_1\pi_2$ is in g.

(2) Multiplication is associative. $(\pi_1\pi_2)\pi_3 = \pi_1(\pi_2\pi_3)$.

(3) There exists an identity element, (e), in g such that for all π in g, $(e)\pi = \pi(e) = \pi$.

(4) For each π in g, there exists a unique inverse, π^{-1}, such that $\pi\pi^{-1} = \pi^{-1}\pi = (e)$. (8.2)

The group denoted by S_n is the group formed by taking the set of all permutations of n objects. It contains n! elements. One would then say that the Hamiltonian in (8.1) has S_3 as its symmetry group since it is invariant to all S_3 operators.

Given the group rules in (8.2), it is possible to write a multiplication table, Table 8.1, for the two elements of S_2 and the six elements of S_3. These tables are read as (x)(y) = table value. The meaning of these operators can be best explained by example. (12) is read as "replace index 1 by index 2 and replace index 2 by index 1", or, $1 \to 2 \to 1$. (123) means "replace 1 by 2, 2 by 3, and 3 by 1", or $1 \to 2 \to 3 \to 1$. Suppose one has a function in three variables, $f(x_1x_2,x_3) = x_1^2x_2^3x_3$, then,

$$(12)f(x_1,x_2,x_3) = f(x_2,x_1,x_3) = x_2^2x_1^3x_3 \quad .$$

Also,

$$(123)f(x_1,x_2,x_3) = f(x_2,x_3,x_1) = x_2^2x_3^3x_1$$

Table 8.1. Multiplication tables of S_2 and S_3

S_2:

(x) \ (y)	(e)	(12)
(e)	(e)	(12)
(12)	(12)	(e)

S_3:

(x) \ (y)	(e)	(12)	(23)	(13)	(123)	(132)
(e)	(e)	(12)	(23)	(13)	(123)	(132)
(12)	(12)	(e)	(123)	(132)	(23)	(13)
(23)	(23)	(132)	(e)	(123)	(13)	(12)
(13)	(13)	(123)	(132)	(e)	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(132)	(e)
(132)	(132)	(23)	(13)	(12)	(e)	(123)

and

$$(12)(123)f(x_1,x_2,x_3) = (12)f(x_2,x_3,x_1)$$
$$= f(x_1,x_3,x_2) = (23)f(x_1,x_2,x_3) \quad .$$

The last example shows explicitly the result from Table 8.1, (12)(123) = (23). An element (ij) is called a transposition and every element of S_n may be written as a product of transpositions. Note that within the dashed box in S_3 of Table 8.1, the multiplication table for S_2 occurs. Thus, S_2 is contained in S_3, or in more rigorous language, S_2 is a subgroup of S_3. This result is expected since the set of all permutations of three objects contains as a subset all the permutations of just two of these objects. In general, S_k is a subgroup of S_n for $k \leq n$. This result suggests that one can use recursion relations to generate properties of S_n once those of S_{n-1} are known. This will be seen in some properties of S_3 as framed in terms of those of S_2.

In order to classify the transformation properties of a linear vector space, $F = \{f_n: \ n=1, \ldots, N\}$, under a group G, a matrix representation of G operating on the set F is needed. That is, the matrices $D(\pi)$ must be found such that

$$\pi f_n = \sum_j f_j D_{jn}(\pi) \quad . \tag{8.3}$$

The matrix multiplication is done in this manner to ensure that

$$\pi_1\pi_2 F = FD(\pi_1\pi_2) = FD(\pi_1)D(\pi_2)$$

which then guarantees that the matrix representation of the operators

preserves the multiplication table of G. The matrices $D(\pi)$ are unitary for all groups with a finite number of elements. The form of the matrices will depend upon the set F and by applying a unitary transformation U to the set F, one can correspondingly change the form of the matrices. The transformed matrices will be given as $D'(\pi) = U^{-1}D(\pi)U$ and will follow the group multiplication table on the new set $F' = UF$. Using a suitable unitary transformation, one may then simplify these matrices as far as possible to block diagonal form. In fact, this block diagonal form can be chosen merely from the properties of G. The projection operators associated with this block diagonal form then contain information about the corresponding unitary transformation and as such will give the appropriate classification of the set of functions F under the group G. This block diagonal form gives all the matrices of G as

$$D(\pi) = \begin{bmatrix} D^1(\pi) & 0 & \\ 0 & D^2(\pi) & \\ & & \ddots \end{bmatrix}$$

where for all π in G, each $D^j(\pi)$ is of dimension $H_j \times H_j$. The $D^j(\pi)$ follow the group multiplication table separately for each j value. This then partitions F into sets F^j of dimension H_j through the unitary transformation such that each element of a given set F^j transforms into a linear combination of elements in that same set F^j under each element of G. When the blocks are as small as possible, each set of matrices $\{D^j(\pi)$: all π in G$\}$ is defined as an irreducible representation, usually called an irrep. Thus, j will be an irrep label and each irrep j will

have an H_j dimensional basis of functions, the set F^j. Since the group has a finite number of elements, the number of unique irreps (unique in that they are not related by a unitary transformation) is finite. That is, no matter how large the set of functions F, the set can always be partitioned into sets which transform in at most N_G different ways, where N_G depends on the group G.

For the groups S_2 and S_3, the Yamanouci labeling and related matrices are presented here. S_2 has two irreps, labeled as [2], which has one basis function, $|11\rangle$, and $[1^2]$, which has one basis function, $|21\rangle$. S_3 has three irreps. [3] has one basis function, $|111\rangle$. [2,1] has two dimensional basis, $|211\rangle$ and $|121\rangle$. $[1^3]$ has one basis function, $|321\rangle$. The matrix representations for S_2 and S_3 are given in Table 8.2. For S_3, only the matrices for (e), (12), and (23) are given; the rest may be found by using Table 8.1. The basis function labels are used as row and column labels for the matrices. As an example, the components of the $D^{[2,1]}(23)$ matrix are

$$D^{[2,1]}_{211,211}(23) = -\frac{1}{2} \quad , \quad D^{[2,1]}_{211,121}(23) = \frac{\sqrt{3}}{2} \quad ,$$

$$D^{[2,1]}_{121,211}(23) = \frac{\sqrt{3}}{2} \quad , \text{ and } \quad D^{[2,1]}_{121,121}(23) = \frac{1}{2} \quad .$$

Then, from Eq. (8.3), one may write down the following transformations

$$(23)\,|211\rangle = -\frac{1}{2}\,|211\rangle + \frac{\sqrt{3}}{2}\,|121\rangle$$

and

Table 8.2. Matrices for the S_2 and S_3 irreps

S_2:

Irrep	Basis Function	(e)	(12)
[2]	|11>	1	1
$[1^2]$	|21>	1	-1

S_3:

Irrep	Basis Function	(e)	(12)	(23)
[3]	|111>	1	1	1
[2,1]	|211>	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$
	|121>			
$[1^3]$	|321>	1	-1	-1

$$(23)\,|121\rangle = \frac{\sqrt{3}}{2}\,|211\rangle + \frac{1}{2}\,|121\rangle \quad ,$$

Thus, for the group S_2, $|11\rangle$ is symmetric under particle interchange and $|21\rangle$ is antisymmetric. For the group S_3, $|111\rangle$ is symmetric and $|321\rangle$ is antisymmetric. Under the subgroup S_2 operating on the indices 1 and 2 only, $|111\rangle$ and $|211\rangle$ are symmetric while $|121\rangle$ and $|321\rangle$ are antisymmetric. This gives a feeling for the properties of the S_2 irreps as contained in the S_3 irreps. As an analogy which is familiar, in angular momentum theory, the angular momentum label of a wavefunction, J, serves as the irrep label and the basis functions for the irrep J are the 2J+1 states $\{|JM\rangle: M = -J, -J+1, \ldots, J\}$. The group operators are all the rotations of the system and the matrices of the irreps are called rotation matrices. (See Appendix A for a review of angular momentum.)

Given the explicit form of the above irreps, it is now known how a given set of functions must transform under the group operators after applying a unitary transformation to the set. To get these transformed functions from the given set of functions $F = \{f_n: n = 1, \ldots, N\}$, projection operators are used. They are defined as

$$P^j_{k\ell} = \frac{H_j}{G_D} \sum_{\text{all } \pi \text{ in } G} D^j_{k\ell}(\pi)^* \pi \quad . \tag{8.4}$$

j is the irrep label, k and ℓ are the irrep basis indices, H_j is the irrep dimension, and G_D is the group dimension. If the set F forms what is called the regular representation of the group G, then there are G_D different projection operators from which the transformed set is completely determined. However, for the purpose of this thesis, it will

be assumed that F does not form the regular representation. In this case, the projection operators may still be used but they do not completely determine the transformed functions. Here, one must fix the ℓ value for each irrep j in (8.4) and then use only the resulting set of operators on F. The explicit method will be shown in the following. Fixing the value of ℓ in S_2 and S_3 is not necessary except for the [2,1] irrep; the only irrep of dimension greater than one. Here, it is fixed at 211. The resulting set of projection operators are given in Table 8.3.

Before proceeding to the general method of operating on the set F, one must examine two important relations which the projection operators satisfy. They are

$$\pi P^j_{k\ell} = \sum_m D^j_{mk}(\pi) P^j_{m\ell} \tag{8.5}$$

and

$$P^j_{k\ell} P^{j'}_{k'\ell'} = \delta_{jj'}[\delta_{\ell k'} P^j_{k\ell'}] \quad . \tag{8.6}$$

In comparing (8.5) with the form of (8.3), it is seen that for some function f in F, if $P^j_{k\ell} f \neq 0$, then $P^j_{k\ell} f$ transforms as the $|k\rangle$ basis function of the j irrep. This is why these are called projection operators. If one fixed $\ell = k$ in (8.5), then (8.6) implies

$$P^j_{k'k} P^j_{kk} f = P^j_{k'k} f$$

and this is nonzero if and only if $P^j_{kk} f$ was nonzero. Thus, $P^j_{k'k} P^j_{kk} f = P^j_{k'k} f$ transforms as the $|k'\rangle$ basis function of the j irrep, of which

Table 8.3. Projection operators of S_2 and S_3

S_2:	$P^{[2]}_{11,11}$	$= 1/2[(e) + (12)]$
	$P^{[1^2]}_{21,21}$	$= 1/2[(e) - (12)]$
S_3:	$P^{[3]}_{111,111}$	$= 1/6[(e) + (12) + (23) + (13) + (123) + (132)]$
	$P^{[2,1]}_{211,211}$	$= 1/3[(e) + (12) - 1/2[(23) + (13) + (123) + (132)]]$
	$P^{[2,1]}_{121,211}$	$= 1/3[\sqrt{3}/2[(23) - (13) - (123) + (132)]]$
	$P^{[1^3]}_{321,321}$	$= 1/6[(e) - (12) - (23) - (13) + (123) + (132)]$

$|k\rangle = P^j_{kk} f$ is also a member. For example, if $P^{[2,1]}_{211,211} f \neq 0$, then $P^{[2,1]}_{121,211} f \neq 0$ and $P^{[2,1]}_{211,211} f$ and $P^{[2,1]}_{121,211} f$ transform as the [2,1] irrep basis functions of Table 8.3. That is, in matrix multiplication

$$\pi(P^{[2,1]}_{211,211} f,\ P^{[2,1]}_{121,211} f) = (P^{[2,1]}_{211,211} f,\ P^{[2,1]}_{121,211} f) D^{[2,1]}(\pi) \quad . \quad (8.7)$$

The general procedure is to choose an irrep j and for a fixed value k, operate on each function f_n in F with $P^j_{k'k}$ for all k' values in the j irrep. Then, repeat for the next irrep. For each irrep, this gives a new set of functions given formally as

$$F^j = \{P^j_{k'k} f_n:\ n = 1, \ldots, N \text{ and all } k' \text{ in } j\} \quad . \quad (8.8)$$

Some of these functions may be zero for all k' given a particular value of n. Then, taking only the nonzero functions from F^j gives the sets

$$F^j_l = \{P^j_{k'k} f_l:\ \text{all } k' \text{ in } j\} \quad (8.9)$$

where l takes at most N different values and for each l, F^j_l forms a basis of functions for the j irrep. Hence, many occurrences of the j irrep will result in general. However, more processing must be done since these irreps in (8.9) are not necessarily orthogonal in the sense that for $l \neq l'$, $P^j_{k'k} f_l$ and $P^j_{k'k} f_{l'}$ are not necessarily orthogonal. To achieve orthogonal occurrences of the same type of irrep, the Gram-Schmit orthonormalization procedure may be used. It is applied separately to each set of functions

$$\{P^j_{k'k} f_l:\ \text{all } l \text{ such that } P^j_{k'k} f_l \neq 0\}$$

where j and k' are fixed. This gives the orthonormal set for each irrep j and basis function $|k'\rangle$

$$\{|k',i\rangle: \quad i = 1, \ldots, K_j\} \tag{8.10}$$

where K_j is the number of nonzero, orthogonal irreps derived from (8.9). One then says that $|k',i\rangle$ is the $|k'\rangle$ basis function in the ith occurrence of the j irrep. For the j irrep, the Gram-Schmit procedure also guarantees that the set

$$j^i = \{|k',i\rangle: \quad \text{all } k' \text{ in the } j \text{ irrep}\} \tag{8.11}$$

is a proper basis for the j irrep in the sense of (8.3) and (8.7). Hence, (8.11) is called the ith occurrence of the j irrep. For example, suppose $F = \{f_1, f_2, f_3\}$ is an orthonormal set of three particle wavefunctions and all possible [2,1] irreps are to be extracted from F. Following the previously outlined method, the set

$$F^{[2,1]} = \{P^{[2,1]}_{211,211} f_i, P^{[2,1]}_{121,211} f_i: \quad i = 1,2,3\}$$

of (8.8) is determined first. Suppose that $P^{[2,1]}_{211,211} f_2$ and $P^{[2,1]}_{121,211} f_2$ are zero, then only the sets

$$F^{[2,1]}_l = \{P^{[2,1]}_{211,211} f_l, P^{[2,1]}_{121,211} f_l: \quad l = 1,3\} \tag{8.12}$$

of (8.8) need to be processed through the Gram-Schmit procedure. If all the states in (8.12) are normalized, then Gram-Schmit gives the orthogonal irreps of (8.11)

$$[2,1]^1 = \{|211,1\rangle = P^{[2,1]}_{211,211} f_1,\ |121,1\rangle = P^{[2,1]}_{121,211} f_1\}$$

and

$$[2,1]^2 = \{|211,2\rangle = N_{211}[-\langle 211,1|P^{[2,1]}_{211,211} f_3\rangle|211,1\rangle + P^{[2,1]}_{211,211} f_3],$$

$$|121,2\rangle = N_{121}[-\langle 121,1|P^{[2,1]}_{121,211} f_3\rangle|121,1\rangle + P^{[2,1]}_{121,211} f_3]\}$$

where N_{211} and N_{121} are normalization factors and $\langle i|j\rangle$ is the dot product of the states $|i\rangle$ and $|j\rangle$. The second occurrence, $[2,1]^2$, may be zero. In this case, $[2,1]^1$ is the only occurrence of the [2,1] irrep in F. This is the procedure used in this thesis.

Another use of these projection operators comes from combining an irrep in one space, j, with an irrep in another space, j', and extracting from that an irrep j'' in simultaneous transformations on the two spaces. For example, given a three particle irrep [2,1], in space coordinates and a [2,1] irrep in spin coordinates, can a [3] irrep under simultaneous permutations of space and spin indices be found and if so, what is its explicit form in terms of the [2,1] space and [2,1] spin functions? These can be found by applying the projection operators to all binary products of basis functions; one from each space. For S_2 and S_3, irreps from two spaces may be combined, or coupled, as given in Table 8.4. The unprimed space is coupled to the primed space and the double-primed space is extracted from that. This is given as $([j] \times [j'])[j'']$ in the table. An explicit form of the j'' basis functions follows to the right of this irrep coupling. As an example, consider the extraction of the [3]'' irrep from the [2,1] and [2,1]' irreps. Applying $P^{[3]}_{111,111}$ to the

Table 8.4. Coupling irreps from S_2 or S_3 to get overall S_2 or S_3 irreps

S_2:	$([2] \times [2]')[2]''$	:	$\|11\rangle'' = \|11\rangle\|11\rangle'$
	$([1^2] \times [1^2]')[2]''$	:	$\|11\rangle'' = \|21\rangle\|21\rangle'$
	$([2] \times [1^2]')[1^2]''$	:	$\|21\rangle'' = \|11\rangle\|21\rangle'$
S_3:	$([3] \times [3]')[3]''$	:	$\|111\rangle'' = \|111\rangle\|111\rangle'$
	$([3] \times [2,1]')[2,1]''$	:	$\|211\rangle'' = \|111\rangle\|211\rangle'$ $\|121\rangle'' = \|111\rangle\|121\rangle'$
	$([3] \times [1^3]')[1^3]''$	:	$\|321\rangle'' = \|111\rangle\|321\rangle'$
	$([2,1] \times [2,1]')[3]''$	:	$\|111\rangle'' = 1/\sqrt{2}\,[\|211\rangle\|211\rangle' + \|121\rangle\|121\rangle']$
	$([2,1] \times [2,1]')[2,1]''$	:	$\|211\rangle'' = 1/\sqrt{2}\,[\|211\rangle\|211\rangle' - \|121\rangle\|121\rangle']$ $\|121\rangle'' = -1/\sqrt{2}\,[\|211\rangle\|121\rangle' + \|121\rangle\|211\rangle']$
	$([2,1] \times [2,1]')[1^3]''$	:	$\|321\rangle'' = 1/\sqrt{2}\,[\|211\rangle\|121\rangle' - \|121\rangle\|211\rangle']$
	$([2,1] \times [1^3]')[2,1]''$	:	$\|211\rangle'' = -\|121\rangle\|321\rangle'$ $\|121\rangle'' = \|211\rangle\|321\rangle'$
	$([1^3] \times [1^3]')[3]''$	:	$\|111\rangle'' = \|321\rangle\|321\rangle'$

product wavefunctions $|211\rangle|211\rangle'$ gives by Table 8.3, where all operators in $P^{[3]}_{111,111}$ permute indices simultaneously in the unprimed and primed spaces,

$$\begin{aligned} P^{[3]}_{111,111}|211\rangle|211\rangle' &= \tfrac{1}{6}\,[|211\rangle|211\rangle' + |211\rangle|211\rangle' \\ &+ (-\tfrac{1}{2}|211\rangle + \tfrac{\sqrt{3}}{2}|121\rangle)(-\tfrac{1}{2}|211\rangle' + \tfrac{\sqrt{3}}{2}|121\rangle') \\ &+ (-\tfrac{1}{2}|211\rangle - \tfrac{\sqrt{3}}{2}|121\rangle)(-\tfrac{1}{2}|211\rangle' - \tfrac{\sqrt{3}}{2}|121\rangle') \\ &+ (-\tfrac{1}{2}|211\rangle - \tfrac{\sqrt{3}}{2}|121\rangle)(-\tfrac{1}{2}|211\rangle' - \tfrac{\sqrt{3}}{2}|121\rangle') \\ &+ (-\tfrac{1}{2}|211\rangle + \tfrac{\sqrt{3}}{2}|121\rangle)(-\tfrac{1}{2}|211\rangle' + \tfrac{\sqrt{3}}{2}|121\rangle')] \\ &= \tfrac{1}{2}\,[|211\rangle|211\rangle' + |121\rangle|121\rangle'] \quad . \end{aligned}$$

Normalizing yields the result of Table 8.4. This table is analogous to the coupling of two angular momenta to give a definite value of the total angular momentum. For instance, $[2,1]\times[2,1]'$ gives three irreps, $[3]''$, $[2,1]''$, and $[1^3]''$. This is similar to $\underset{\sim}{j}_1 + \underset{\sim}{j}_2 = \underset{\sim}{J}$. If $j_1 = 1$, $j_2 = 2$, then J may be equal to 1, 2, or 3. The explicit form of the functions in the space coupled to total angular momentum J are given using Clebsch-Gordan coefficients and product wavefunctions as in (7.7), analogous to the double-primed functions of Table 8.4.

IX. APPENDIX C: DERIVATION OF A SPECIAL CASE OF THE TALMI BRACKETS

Suppose that a system may be represented by two particles in harmonic oscillator potentials where the oscillator frequency, ω, associated with each particle is the same. The appropriate Hamiltonian, for possibly unequal mass particles, is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}\omega^2(m_1 r_1^2 + m_2 r_2^2) \tag{9.1}$$

where m_k is the mass of particle k, $\underset{\sim}{p}_k = -i\underset{\sim}{\nabla}_k$ is the momentum conjugate to $\underset{\sim}{r}_k$, and $\hbar = 1$. By (7.7), the wavefunctions of the system, coupled to total orbital angular momentum L and projection M, are

$$|(N_1L_1N_2L_2)LM\rangle = \sum_{M_1M_2} C(L_1L_2L;\ M_1M_2M)\,|N_1L_1M_1\rangle|N_2L_2M_2\rangle \tag{9.2}$$

where $\underset{\sim}{L} = \underset{\sim}{L}_1 + \underset{\sim}{L}_2$. $|NLM\rangle$ is a single particle, harmonic oscillator state and is represented in spherical coordinates as

$$\phi_{NLM} = \left[\frac{2N!}{b^3\Gamma(N+L+3/2)}\right]^{1/2} \left(\frac{r}{b}\right)^L L_N^{L+\frac{1}{2}}\left(\frac{r^2}{b^2}\right) e^{-\frac{1}{2}\frac{r^2}{b^2}} Y_{LM}(\hat{r}) \quad . \tag{9.3}$$

$L_N^{L+\frac{1}{2}}$ is an associated Laguerre polynomial (20) given as

$$L_N^{L+\frac{1}{2}}(x) = \sum_{K=0}^{N} \frac{\Gamma(N+L+3/2)}{\Gamma(K+L+3/2)} \binom{N}{K} \frac{1}{N!}(-x)^K \tag{9.4}$$

and Y_{LM} is a spherical harmonic function (25). Also, $b = (m\omega)^{-1/2}$ is the oscillator length parameter. One may wish to apply a coordinate

transformation, π, to the wavefunctions $\phi_{N_1L_1M_1}(\underset{\sim}{r}_1)$ and $\phi_{N_2L_2M_2}(\underset{\sim}{r}_2)$ such that

$$\pi \begin{bmatrix} \underset{\sim}{r}_1 \\ \underset{\sim}{r}_2 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \underset{\sim}{r}_1 \\ \underset{\sim}{r}_2 \end{bmatrix} \tag{9.5}$$

while preserving the harmonic oscillator form of the Hamiltonian. That is, in terms of $\underset{\sim}{R}_1$, $\underset{\sim}{R}_2$, $\underset{\sim}{P}_1$, and $\underset{\sim}{P}_2$, where $\underset{\sim}{P}_1$ is the momentum conjugate to $\underset{\sim}{R}_1$ and $\underset{\sim}{P}_2$ that conjugate to $\underset{\sim}{R}_2$, H is recast as

$$H = \frac{P_1^2}{2\mu_1} + \frac{P_2^2}{2\mu_2} + \frac{1}{2}\omega^2(\mu_1 R_1^2 + \mu_2 R_2^2) \quad .$$

μ_1 and μ_2 may be thought of as masses associated with the transformed variables $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$. The constraint of preserving the oscillator form of the Hamiltonian implies that

$$\frac{m_1}{m_2} = -\frac{ac}{bd}$$

which gives the general results

$$\mu_1 = (m_1 d^2 + m_2 c^2)/(ad - bc)^2$$

and

$$\mu_2 = (m_1 b^2 + m_2 a^2)/(ad - bc)^2 \quad .$$

For example, this type of transformation is encountered in the transformation to relative and center of mass coordinates. In this case,

$$\underset{\sim}{R}_1 = \frac{1}{m_1 + m_2} (m_1 \underset{\sim}{r}_1 + m_2 \underset{\sim}{r}_2)$$

and

$$\underset{\sim}{R}_2 = \underset{\sim}{r}_1 - \underset{\sim}{r}_2$$

which implies

$$\mu_1 = m_1 + m_2$$

and

$$\mu_2 = \frac{m_1 m_2}{m_1 + m_2} \quad .$$

$\underset{\sim}{R}_1$ is the center of mass coordinate and $\underset{\sim}{R}_2$ is the relative coordinate. The type of transformation used in this thesis is somewhat different than the transformation to relative and center of mass coordinates. The one used here involves the permutation of three particles in a system of Jacobi coordinates. In this case, $m_1 = m_2$ and the transformation is

$$\begin{bmatrix} \underset{\sim}{\rho}' \\ \underset{\sim}{\lambda}' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\rho} \\ \underset{\sim}{\lambda} \end{bmatrix}$$

which is orthogonal. The Hamiltonian,

$$H = \frac{1}{2m} (P_\lambda^2 + P_\rho^2) + \frac{1}{2} m\omega^2 (\lambda^2 + \rho^2)$$

becomes

$$H = \frac{1}{2m} (P_{\lambda'}^2 + P_{\rho'}^2) + \frac{1}{2} m\omega^2 (\lambda'^2 + \rho'^2) \quad .$$

Thus, the transformation used in this appendix will be given by (9.5) where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal and $m_1 = m_2$. Applying (9.5) to (9.2) then gives

$$\pi|(N_1L_1N_2L_2)LM\rangle = \pi \sum_{M_1M_2} C(L_1L_2L;\, M_1M_2M)\phi_{N_1L_1M_1}(\underset{\sim}{r}_1)\phi_{N_2L_2M_2}(\underset{\sim}{r}_2)$$

$$= \sum_{M_1M_2} C(L_1L_2L;\, M_1M_2M)\phi_{N_1L_1M_1}(\underset{\sim}{R}_1)\phi_{N_2L_2M_2}(\underset{\sim}{R}_2) \quad . \quad (9.6)$$

The goal is then to write the right side of (9.6) in terms of the original, complete set of coupled states in the variables $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ of Eqs. (9.2) and (9.3). Explicitly, this set is

$$\{|(n_1\ell_1n_2\ell_2)\ell m\rangle:\ n_1,n_2,\ell_1,\ell_2 = 0,1,2,\ \ldots \quad \text{and}$$

$$\ell \geq |\ell_1-\ell_2|,\ \ell \leq \ell_1+\ell_2,\ m = -\ell,-\ell+1,\ \ldots,\ \ell\} \quad .$$

Then an expression must be found for the transformation coefficient, known as a Talmi bracket, $\langle N_1L_1N_2L_2LM|n_1\ell_1n_2\ell_2\ell m\rangle$ such that

$$\sum_{M_1M_2} C(L_1L_2L;\, M_1M_2M)\phi_{N_1L_1M_1}(\underset{\sim}{R}_1)\phi_{N_2L_2M_2}(\underset{\sim}{R}_2)$$

$$= \sum_{n_1n_2\ell_1\ell_2\ell m} \langle N_1L_1N_2L_2LM|n_1\ell_1n_2\ell_2\ell m\rangle \sum_{m_1m_2} C(\ell_1\ell_2\ell;m_1m_2m)$$

$$\times\, \phi_{n_1\ell_1m_1}(\underset{\sim}{r}_1)\phi_{n_2\ell_2m_2}(\underset{\sim}{r}_2) \quad . \quad (9.7)$$

The following is one method of deriving the Talmi bracket. The closed form result has been determined previously by Baranger and Davies (27), by Talman (28), and by Cooper (29). This derivation will basically follow Cooper's method. Talmi (30) first determined these by brute force and later Moshinsky (31) found recurrence relations for them.

To begin, orthonormality of $|(n_1\ell_1 n_2\ell_2)\ell m\rangle$ as applied to (9.7) gives

$$\langle N_1 L_1 N_2 L_2 LM | n_1\ell_1 n_2\ell_2 \ell m\rangle = \sum_{M_1 M_2 m_1 m_2} C(L_1 L_2 L; M_1 M_2 M) C(\ell_1 \ell_2 \ell; m_1 m_2 m)$$
$$\times \int\int d^3r_1 d^3r_2 \phi^*_{n_1\ell_1 m_1}(\underset{\sim}{r}_1)\phi^*_{n_2\ell_2 m_2}(\underset{\sim}{r}_2)$$
$$\times \phi_{N_1 L_1 M_1}(\underset{\sim}{R}_1)\phi_{N_2 L_2 M_2}(\underset{\sim}{R}_2) \quad . \tag{9.8}$$

Making the substitution $\underset{\sim}{r}_1 \to b\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2 \to b\underset{\sim}{r}_2$ in (9.8) removes all the b factors in the wavefunctions. Expanding the wavefunctions using (9.3) and expanding $L_{N_1}^{L_1+\frac{1}{2}}(R_1^2)$ and $L_{N_2}^{L_2+\frac{1}{2}}(R_2^2)$ using (9.4) gives

$$\langle N_1 L_1 N_2 L_2 LM | n_1\ell_1 n_2\ell_2 \ell m\rangle = \sum_{M_1 M_2 m_1 m_2} C(L_1 L_2 L; M_1 M_2 M) C(\ell_1 \ell_2 \ell; m_1 m_2 m)$$
$$\times 4n_1! n_2! \left[\frac{G(N_1,L_1)G(N_2,L_2)}{G(n_1,\ell_1)G(n_2,\ell_2)}\right]^{1/2}$$
$$\times \sum_{K_1 K_2} \frac{(-1)^{K_1+K_2}}{G(K_1,L_1)G(K_2,L_2)(N_1-K_1)!(N_2-K_2)!}$$
$$\times \int\int d^3r_1 d^3r_2 r_1^{\ell_1} r_2^{\ell_2} L_{n_1}^{\ell_1+\frac{1}{2}}(r_1^2)\, L_{n_2}^{\ell_2+\frac{1}{2}}(r_2^2)\, e^{-(r_1^2+r_2^2)}$$
$$\times R_1^{2K_1+L_1} R_2^{2K_2+L_2}\, Y^*_{\ell_1 m_1}(r_1)\, Y^*_{\ell_2 m_2}(\hat{r}_2)\, Y_{L_1 M_1}(\hat{R}_1)\, Y_{L_2 M_2}(\hat{R}_2) \quad , \tag{9.9}$$

where

$$G(a,b) = a!\Gamma(a+b+3/2) \quad . \tag{9.10}$$

To perform the integration in (9.9), $R_1^{2K_1+L_1} Y_{L_1M_1}(\hat{R}_1)$ and $R_2^{L_2+2K_2} Y_{L_2M_2}(\hat{R}_2)$ need to be rewritten in terms of functions in the $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ variables where $\underset{\sim}{R}_1 = a\underset{\sim}{r}_1 + b\underset{\sim}{r}_2$ and $\underset{\sim}{R}_2 = c\underset{\sim}{r}_1 + d\underset{\sim}{r}_2$. Starting with the equality

$$e^{i\underset{\sim}{Q}\cdot\underset{\sim}{R}_1} = e^{ia\underset{\sim}{Q}\cdot\underset{\sim}{r}_1} e^{ib\underset{\sim}{Q}\cdot\underset{\sim}{r}_2}$$

and expanding the exponentials using the formula (23)

$$e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}} = 4\pi \sum_{L=0}^{\infty} \sum_{M=-L}^{L} i^L j_L(pr)\, Y_{LM}(\hat{p})\, Y^*_{LM}(\hat{r})$$

gives

$$\sum_{L_1M_1} i^{L_1} j_{L_1}(QR_1)\, Y_{L_1M_1}(\hat{Q})\, Y^*_{L_1M_1}(\hat{R}_1) = 4\pi \sum_{\ell_1'\ell_2'm_1'm_2'} i^{\ell_1'+\ell_2'} j_{\ell_1'}(aQr_1)$$

$$\times\, j_{\ell_2'}(bQr_2)\, Y_{\ell_1'm_1'}(\hat{Q})\, Y^*_{\ell_1'm_1'}(\hat{r}_1)\, Y_{\ell_2'm_2'}(\hat{Q})\, Y^*_{\ell_2'm_2'}(\hat{r}_2)\,, \quad (9.11)$$

where j_ℓ is a spherical Bessel function (20). Multiplying both sides of (9.11) by $Y^*_{L_1'M_1'}(\hat{Q})$, integrating over the solid angle Ω in the variable $\hat{Q}$, using the orthonormality of $Y_{L_1M_1}(\hat{Q})$ on the left hand side of (9.11), and using (7.16) on the right hand side gives, after taking the complex conjugate,

$$j_{L_1}(QR_1)\, Y_{L_1M_1}(\hat{R}_1) = (4\pi)^{1/2} \sum_{\ell_1'\ell_2'm_1'm_2'} (-1)^{A_1} j_{\ell_1'}(aQr_1)\, j_{\ell_2'}(bQr_2)$$

$$\times\, Y_{\ell_1'm_1'}(\hat{r}_1)\, Y_{\ell_2'm_2'}(\hat{r}_2) \left[\frac{\hat{\ell}_1'\hat{\ell}_2'}{\hat{L}_1} \right]$$

$$\times C(\ell_1'\ell_2'L_1;\ m_1'm_2'M_1)C(\ell_1'\ell_2'L_1;\ 000) \quad . \qquad (9.12)$$

In (9.12), $\hat{\ell} = \sqrt{2\ell+1}$ and $2A_1 = \ell_1' + \ell_2' - L_1$ is an even integer because $C(\ell_1'\ell_2'L_1;\ 000)$ is zero unless $\ell_1' + \ell_2' + L_1$ is even. Then, substituting the power series expansion of the spherical Bessel function,

$$j_L(x) = \frac{\sqrt{\pi}}{2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{L+2\nu}}{G(\nu,L)}$$

into (9.12), multiplying by Q^{-L_1}, and equating coefficients of equal powers of Q gives

$$R_1^{L_1+2K_1} Y_{L_1M_1}(\hat{R}_1) = \pi G(K_1,L_1) \sum_{\ell_1'\ell_2'm_1'm_2'} Y_{\ell_1'm_1'}(\hat{r}_1)\, Y_{\ell_2'm_2'}(\hat{r}_2) \left[\frac{\hat{\ell}_1'\hat{\ell}_2'}{\hat{L}_1}\right]$$

$$\times C(\ell_1'\ell_2'L_1;\ m_1'm_2'M_1)C(\ell_1'\ell_2'L_1;\ 000)$$

$$\times \sum_{\nu_1,\nu_2=0}^{\infty} \delta_{\nu_1+\nu_2+A_1,K_1} \frac{(ar_1)^{\ell_1'+2\nu_1} (br_2)^{\ell_2'+2\nu_2}}{G(\nu_1,\ell_1')G(\nu_2,\ell_2')} \quad . \qquad (9.13)$$

This is the formula needed and then clearly one may also write

$$R_2^{L_2+2K_2} Y_{L_2M_2}(\hat{R}_2) = \pi G(K_2,L_2) \sum_{\ell_1''\ell_2''m_1''m_2''} Y_{\ell_1''m_1''}(\hat{r}_1)\, Y_{\ell_2''m_2''}(\hat{r}_2) \left[\frac{\hat{\ell}_1''\hat{\ell}_2''}{\hat{L}_2}\right]$$

$$\times C(\ell_1''\ell_2''L_2;\ m_1''m_2''M_2)C(\ell_1''\ell_2''L_2;\ 000)$$

$$\times \sum_{\mu_1,\mu_2=0} \delta_{\mu_1+\mu_2+A_2,K_2} \frac{(cr_1)^{\ell_1''+2\mu_1} (dr_2)^{\ell_2''+2\mu_2}}{G(\mu_1,\ell_1'')G(\mu_2,\ell_2'')} \quad , \qquad (9.14)$$

where $2A_2 = \ell_1'' + \ell_2'' - L_2$. Substituting (9.13) and (9.14) into (9.9), integrating over the solid angles Ω_1 and Ω_2 by using (7.16), and integrating over r_1 and r_2 using the easily derived relation

$$\int_0^\infty dx\, e^{-x} x^{\nu+\alpha} L_n^\alpha(x) = (-1)^n \binom{\nu}{n} \Gamma(\nu + \alpha + 1)$$

gives

$$\langle N_1 L_1 N_2 L_2 LM | n_1 \ell_1 n_2 \ell_2 \ell m \rangle = \sum_{M_1 M_2 m_1 m_2} C(L_1 L_2 L; M_1 M_2 M) C(\ell_1 \ell_2 \ell; m_1 m_2 m)$$

$$\times \frac{\pi}{4} (-1)^{n_1+n_2} n_1! n_2! \left[\frac{G(N_1, L_1) G(N_2, L_2)}{G(n_1, \ell_1) G(n_2, \ell_2)} \right]^{1/2} \sum_{K_1 K_2} \frac{(-1)^{K_1+K_2}}{(N_1-K_1)!\,(N_2-K_2)!}$$

$$\times \sum_{\substack{\ell_1' \ell_1'' \ell_2' \ell_2'' \\ m_1' m_1'' m_2' m_2''}} \frac{(\hat{\ell}_1' \hat{\ell}_1'' \hat{\ell}_2' \hat{\ell}_2'')^2}{\hat{L}_1 \hat{L}_2 \hat{\ell}_1 \hat{\ell}_2} C(\ell_1' \ell_2' L_1; m_1' m_2' M_1) C(\ell_1' \ell_2' L_1; 000)$$

$$\times C(\ell_1'' \ell_2'' L_2; m_1'' m_2'' M_2) C(\ell_1'' \ell_2'' L_2; 000) C(\ell_1' \ell_1'' \ell_1; m_1' m_1'' m_1) C(\ell_1' \ell_1'' \ell_1; 000)$$

$$\times C(\ell_2' \ell_2'' \ell_2; m_2' m_2'' m_2) C(\ell_2' \ell_2'' \ell_2; 000) \sum_{\nu_1 \nu_2 \mu_1 \mu_2}$$

$$\times \frac{a^{\ell_1'+2\nu_1} b^{\ell_2'+2\nu_2} c^{\ell_1''+2\mu_1} d^{\ell_2''+2\mu_2} \delta_{\nu_1+\nu_2+A_1, K_1} \delta_{\mu_1+\mu_2+A_2, K_2}}{G(\nu_1, \ell_1') G(\nu_2, \ell_2') G(\mu_1, \ell_1'') G(\mu_2, \ell_2'')}$$

$$\times \Gamma(B_1+\nu_1+\mu_1+\ell_1+3/2) \Gamma(B_2+\nu_2+\mu_2+\ell_2+3/2) \binom{B_1+\nu_1+\mu_1}{n_1} \binom{B_2+\nu_2+\mu_2}{n_2}, \tag{9.15}$$

where $2B_1 = \ell_1' + \ell_1'' - \ell_1$ and $2B_2 = \ell_2' + \ell_2'' - \ell_2$ are also even integers.

Since this transformation leaves the Hamiltonian invariant in form, it will not change the physical observables associated with the wavefunctions in (9.2). Thus, in the expansion of (9.7), the total angular momentum, its projection, and the total energy must be the same on both sides. Explicitly,

$$L = \ell \quad , \tag{9.16}$$

$$M = m \quad \text{where} \quad M = M_1 + M_2 \quad \text{and} \quad m = m_1 + m_2 \quad , \tag{9.17}$$

and

$$2N_1 + L_1 + 2N_2 + L_2 = 2n_1 + \ell_1 + 2n_2 + \ell_2 \tag{9.18}$$

where the total energy of the two particle system is $\omega(2N_1 + L_1 + 2N_2 + L_2 + 3)$. The same is true of the parity of the wavefunctions in (9.7) which implies that

$$(-1)^{L_1+L_2} = (-1)^{\ell_1+\ell_2} \quad . \tag{9.19}$$

Using the two Kronecker delta functions in (9.15) with the fact that K_i is summed from 0 to N_i for $i = 1,2$ and by using Eq. (9.18), it can be shown that

$$K_1 = N_1 \quad , \quad K_2 = N_2 \quad , \quad B_1 + \mu_1 + \nu_1 = n_1 \quad , \quad \text{and} \quad B_2 + \mu_2 + \nu_2 = n_2 \, .$$

The first two equalities eliminate the sums over K_1 and K_2 in (9.15) and the last two will be inserted as Kronecker delta functions. Then, (9.15) becomes

$$\langle N_1 L_1 N_2 L_2 LM | n_1 \ell_1 n_2 \ell_2 \ell m \rangle = \frac{\pi}{4} (-1)^{n_1+n_2+N_1+N_2} [\hat{\ell}_1 \hat{\ell}_2 \hat{L}_1 \hat{L}_2]^{-1}$$

$$\times [G(n_1,\ell_1)G(n_2,\ell_2)G(N_1,L_1)G(N_2,L_2)]^{1/2} \sum_{\ell_1' \ell_1'' \ell_2' \ell_2''} [\hat{\ell}_1' \hat{\ell}_1'' \hat{\ell}_2' \hat{\ell}_2'']^2$$

$$\times C(\ell_1' \ell_2' L_1;\ 000) C(\ell_1'' \ell_2'' L_2;\ 000) C(\ell_1' \ell_1'' \ell_1;\ 000) C(\ell_2' \ell_2'' \ell_2;\ 000)$$

$$\times \sum_{\substack{m_1' m_1'' m_2' m_2'' \\ m_1 m_2 M_1 M_2}} C(\ell_1' \ell_2' L_1;\ m_1' m_2' M_1) C(\ell_1'' \ell_2'' L_2;\ m_1'' m_2'' M_2) C(\ell_1' \ell_1'' \ell_1;\ m_1' m_1'' m_1)$$

$$\times C(\ell_2' \ell_2'' \ell_2;\ m_2' m_2'' m_2) C(L_1 L_2 L;\ M_1 M_2 M) C(\ell_1 \ell_2 \ell;\ m_1 m_2 m) \sum_{\nu_1 \nu_2 \mu_1 \mu_2}$$

$$\times \frac{a^{\ell_1'+2\nu_1} \, b^{\ell_2'+2\nu_2} \, c^{\ell_1''+2\mu_1} \, d^{\ell_2''+2\mu_2}}{G(\nu_1,\ell_1')G(\nu_2,\ell_2')G(\mu_1,\ell_1'')G(\mu_2,\ell_2'')} \, \delta_{A_1+\nu_1+\nu_2,N_2} \, \delta_{A_2+\mu_1+\mu_2,N_2}$$

$$\times \delta_{B_1+\nu_1+\mu_1,n_1} \, \delta_{B_2+\nu_2+\mu_2,n_2} \quad . \tag{9.20}$$

This can be further simplified by using an angular momentum result (32)

$$\sum_{m_1' m_1'' m_2' m_2''} C(\ell_1' \ell_2' L_1;\ m_1' m_2' M_1) C(\ell_1'' \ell_2'' L_2;\ m_1'' m_2'' M_2) C(\ell_1' \ell_1'' \ell_1;\ m_1' m_1'' m_1)$$

$$\times C(\ell_2' \ell_2'' \ell_2;\ m_2' m_2'' m_2) = [\hat{L}_1 \hat{L}_2 \hat{\ell}_1 \hat{\ell}_2] \sum_{JK} C(L_1 L_2 J;\ M_1 M_2 K) C(\ell_1 \ell_2 J;\ m_1 m_2 K)$$

$$\times \begin{Bmatrix} \ell_1' & \ell_2' & L_1 \\ \ell_1'' & \ell_2'' & L_2 \\ \ell_1 & \ell_2 & J \end{Bmatrix} \quad . \tag{9.21}$$

Then, by (7.9), the sum of the remaining Clebsch-Gordan coefficients in (9.20) becomes

$$\sum_{M_1M_2m_1m_2} C(L_1L_2L;\ M_1M_2M)C(L_1L_2J;\ M_1M_2K)C(\ell_1\ell_2\ell;\ m_1m_2m)C(\ell_1\ell_2J;\ m_1m_2K)$$

$$= \delta_{JL}\delta_{J\ell}\delta_{KM}\delta_{Km}$$

and, thus, (9.20) will be simplified to

$$\langle N_1L_1N_2L_2LM|n_1\ell_1n_2\ell_2\ell m\rangle = \delta_{L,\ell}\delta_{M,m}\ \frac{\pi}{4}\ (-1)^{n_1+n_2+N_1+N_2}$$

$$\times\ [G(n_1,\ell_1)G(n_2,\ell_2)G(N_1,L_1)G(N_2,L_2)]^{1/2} \sum_{\ell_1'\ell_1''\ell_2'\ell_2''} [\hat{\ell}_1'\hat{\ell}_1''\hat{\ell}_2'\hat{\ell}_2'']^2$$

$$\times \begin{Bmatrix} \ell_1' & \ell_2' & L_1 \\ \ell_1'' & \ell_2'' & L_2 \\ \ell_1 & \ell_2 & L \end{Bmatrix} C(\ell_1'\ell_2'L_1;\ 000)C(\ell_1''\ell_2''L_2;\ 000)C(\ell_1'\ell_1''\ell_1;\ 000)$$

$$\times\ C(\ell_2'\ell_2''\ell_2;\ 000) \sum_{\nu_1\nu_2\mu_1\mu_2} \frac{a^{\ell_1'+2\nu_1}\ b^{\ell_2'+2\nu_2}\ c^{\ell_1''+2\mu_1}\ d^{\ell_2''+2\mu_2}}{G(\nu_1,\ell_1')G(\nu_2,\ell_2')G(\mu_1,\ell_1'')G(\mu_2,\ell_2'')}$$

$$\times\ \delta_{A_1+\nu_1+\nu_2,N_1}\ \delta_{A_2+\mu_1+\mu_2,N_2}\ \delta_{B_1+\nu_1+\mu_1,n_1}\ \delta_{B_2+\nu_2+\mu_2,n_2} \quad . \qquad (9.22)$$

This is the closed form expression desired for the transformation coefficient $\langle N_1L_1N_2L_2LM|n_1\ell_1n_2\ell_2\ell m\rangle$ and in (9.22), although not stated explicitly,

$$2A_1 = \ell_1' + \ell_2' - L_1 \quad , \quad 2A_2 = \ell_1'' + \ell_2'' - L_2 \quad ,$$

$$2B_1 = \ell_1' + \ell_1'' - \ell_1 \quad \text{and} \quad 2B_2 = \ell_2' + \ell_2'' - \ell_2 \quad .$$

Equations (9.18) and (9.19) also hold in (9.22). Equation (9.18) can be shown, however, to be redundant with one of the Kronecker delta functions in (9.22). The constraint of (9.18) limits the sums in (9.22) to be finite since it can be shown that

$$\ell_1' + \ell_1'' + \ell_2' + \ell_2'' + 2(\mu_1 + \mu_2 + \nu_1 + \nu_2) = 2N_1 + L_1 + 2N_2 + L_2$$

$$= 2n_1 + \ell_1 + 2n_2 + \ell_2 \quad .$$

X. APPENDIX D: MISCELLANEOUS TABLES

This appendix consists of four, lengthy tables, included in this thesis for completeness, but which would have been distracting if included in the main text.

A. Table of the S_3 Irreps in the Space Variables

Table 10.1 is a listing of the space S_3 irreps as extracted from the harmonic oscillator states in the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ variables of Table 3.5. The notation is

$$[m\alpha,\pi> \;=\; [k,i;\; E,L> \;=\; \sum_K C_K^{k,i}\,[K,\pi> \quad ,$$

where $[m\alpha,\pi>$ are the S_3 irrep basis functions. The label m is an index and two sets of indices are used, one for $\pi = +1$ and one for $\pi = -1$, where π is the parity. The label α differentiates the two basis functions of the [2,1] irrep; α = a for the $|211>$ function and α = b for the $|121>$. The label k is the S_3 basis function label and i labels the irrep occurrence for one set of L and E values, where L is the total orbital angular momentum and E is the excitation energy in units of the oscillator frequency, ω. The $[K,\pi>$ are the two-particle, harmonic oscillator states in the $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ variables of Table 3.5 and the $C_K^{k,i}$ are the coefficients of the unitary transformation.

Table 10.1. The S_3 irreps in the space variables

$$
\begin{aligned}
[1,+\rangle &= [(111),1{:}0,0\rangle = (\ 1.0000000)\times[1,+\rangle \\
[2,+\rangle &= [(111),1{:}2,0\rangle = (\ 0.7071068)\times[2,+\rangle \\
&\quad + (\ 0.7071068)\times[3,+\rangle \\
[3a,+\rangle &= [(211),1{:}2,0\rangle = (\ 0.7071067)\times[2,+\rangle \\
&\quad + (-0.7071068)\times[3,+\rangle \\
[3b,+\rangle &= [(121),1{:}2,0\rangle = (-1.0000000)\times[4,+\rangle \\
[4,+\rangle &= [(321),1{:}2,1\rangle = (\ 1.0000000)\times[5,+\rangle \\
[5,+\rangle &= [(111),1{:}2,2\rangle = (\ 0.7071068)\times[6,+\rangle \\
&\quad + (\ 0.7071068)\times[7,+\rangle \\
[6a,+\rangle &= [(211),1{:}2,2\rangle = (\ 0.7071066)\times[6,+\rangle \\
&\quad + (-0.7071068)\times[7,+\rangle \\
[6b,+\rangle &= [(121),1{:}2,2\rangle = (-1.0000000)\times[8,+\rangle \\
[7,+\rangle &= [(111),1{:}4,0\rangle = (\ 0.6123724)\times[9,+\rangle \\
&\quad + (\ 0.6123725)\times[10,+\rangle \\
&\quad + (\ 0.3726780)\times[13,+\rangle \\
&\quad + (\ 0.3333333)\times[14,+\rangle \\
[8,+\rangle &= [(111),2{:}4,0\rangle = (\ 0.6666667)\times[13,+\rangle \\
&\quad + (-0.7453559)\times[14,+\rangle \\
[9a,+\rangle &= [(211),1{:}4,0\rangle = (\ 0.7905694)\times[9,+\rangle \\
&\quad + (-0.4743417)\times[10,+\rangle \\
&\quad + (-0.2886752)\times[13,+\rangle \\
&\quad + (-0.2581989)\times[14,+\rangle \\
[9b,+\rangle &= [(121),1{:}4,0\rangle = (\ 0.3162278)\times[11,+\rangle \\
&\quad + (\ 0.9486833)\times[12,+\rangle \\
[10a,+\rangle &= [(211),2{:}4,0\rangle = (\ 0.6324554)\times[10,+\rangle \\
&\quad + (-0.5773503)\times[13,+\rangle \\
&\quad + (-0.5163977)\times[14,+\rangle \\
[10b,+\rangle &= [(121),2{:}4,0\rangle = (-0.9486833)\times[11,+\rangle \\
&\quad + (\ 0.3162277)\times[12,+\rangle \\
[11a,+\rangle &= [(211),1{:}4,1\rangle = (\ 1.0000000)\times[17,+\rangle \\
[11b,+\rangle &= [(121),1{:}4,1\rangle = (-0.7071068)\times[15,+\rangle \\
&\quad + (\ 0.7071068)\times[16,+\rangle \\
[12,+\rangle &= [(321),1{:}4,1\rangle = (\ 0.7071068)\times[15,+\rangle \\
&\quad + (\ 0.7071068)\times[16,+\rangle \\
[13,+\rangle &= [(111),1{:}4,2\rangle = (\ 0.6123725)\times[18,+\rangle \\
&\quad + (\ 0.6123724)\times[19,+\rangle \\
&\quad + (\ 0.3118048)\times[24,+\rangle \\
&\quad + (\ 0.3118048)\times[25,+\rangle \\
&\quad + (\ 0.2357021)\times[26,+\rangle \\
[14,+\rangle &= [(111),2{:}4,2\rangle = (\ 0.3333337)\times[24,+\rangle \\
&\quad + (\ 0.3333334)\times[25,+\rangle \\
&\quad + (-0.8819170)\times[26,+\rangle \\
[15a,+\rangle &= [(211),1{:}4,2\rangle = (\ 0.7905694)\times[18,+\rangle \\
&\quad + (-0.4743416)\times[19,+\rangle \\
&\quad + (-0.2415230)\times[24,+\rangle \\
&\quad + (-0.2415230)\times[25,+\rangle \\
&\quad + (-0.1825741)\times[26,+\rangle \\
[15b,+\rangle &= [(121),1{:}4,2\rangle = (\ 0.2645751)\times[20,+\rangle \\
&\quad + (\ 0.7937251)\times[21,+\rangle \\
&\quad + (\ 0.1732052)\times[22,+\rangle \\
&\quad + (\ 0.5196155)\times[23,+\rangle
\end{aligned}
$$

Table 10.1. Continued

```
[16a,+> = [(211),2:4,2> =  ( 0.6324556)x[19,+>
                         + (-0.4830459)x[24,+>
                         + (-0.4830459)x[25,+>
                         + (-0.3651483)x[26,+>

[16b,+> = [(121),2:4,2> =  (-0.7937250)x[20,+>
                         + ( 0.2645751)x[21,+>
                         + (-0.5196157)x[22,+>
                         + ( 0.1732052)x[23,+>

[17a,+> = [(211),3:4,2> =  ( 0.7071068)x[24,+>
                         + (-0.7071068)x[25,+>

[17b,+> = [(121),3:4,2> =  ( 0.3872988)x[20,+>
                         + ( 0.3872985)x[21,+>
                         + (-0.5916078)x[22,+>
                         + (-0.5916077)x[23,+>

 [18,+> = [(321),1:4,2> =  ( 0.3872984)x[20,+>
                         + (-0.3872977)x[21,+>
                         + (-0.5916081)x[22,+>
                         + ( 0.5916082)x[23,+>

[19a,+> = [(211),1:4,3> =  ( 1.0000000)x[29,+>

[19b,+> = [(121),1:4,3> =  (-0.7071068)x[27,+>
                         + ( 0.7071066)x[28,+>

 [20,+> = [(321),1:4,3> =  ( 0.7071068)x[27,+>
                         + ( 0.7071068)x[28,+>

 [21,+> = [(111),1:4,4> =  ( 0.6123728)x[30,+>
                         + ( 0.6123726)x[31,+>
                         + ( 0.4999993)x[34,+>

[22a,+> = [(211),1:4,4> =  ( 0.7905697)x[30,+>
                         + (-0.4743417)x[31,+>
                         + (-0.3872977)x[34,+>

[22b,+> = [(121),1:4,4> =  ( 0.3162279)x[32,+>
                         + ( 0.9486833)x[33,+>

[23a,+> = [(211),2:4,4> =  ( 0.6324559)x[31,+>
                         + (-0.7745963)x[34,+>

[23b,+> = [(121),2:4,4> =  (-0.9486833)x[32,+>
                         + ( 0.3162279)x[33,+>

 [24,+> = [(111),1:6,0> =  ( 0.5863020)x[35,+>
                         + ( 0.4797016)x[36,+>
                         + ( 0.1410190)x[39,+>
                         + ( 0.4230570)x[40,+>
                         + ( 0.1507557)x[41,+>
                         + ( 0.4522670)x[42,+>

 [25,+> = [(111),2:6,0> =  ( 0.3370999)x[36,+>
                         + ( 0.5351295)x[39,+>
                         + (-0.3567531)x[40,+>
                         + ( 0.5720775)x[41,+>
                         + (-0.3813851)x[42,+>

 [26,+> = [(111),3:6,0> =  ( 0.5163981)x[39,+>
                         + ( 0.5163979)x[40,+>
                         + (-0.4830458)x[41,+>
                         + (-0.4830455)x[42,+>

[27a,+> = [(211),1:6,0> =  ( 0.8100926)x[35,+>
                         + (-0.3471826)x[36,+>
                         + (-0.1020621)x[39,+>
                         + (-0.3061862)x[40,+>
                         + (-0.1091090)x[41,+>
                         + (-0.3273268)x[42,+>
```

Table 10.1. Continued

```
[27b,+> = [(121),1:6,0> =  ( 0.0944911)x[37,+>
                         + ( 0.8504201)x[38,+>
                         + ( 0.4743415)x[43,+>
                         + ( 0.2070199)x[44,+>

[28a,+> = [(211),2:6,0> =  ( 0.7319251)x[36,+>
                         + (-0.3872983)x[39,+>
                         + (-0.2581989)x[40,+>
                         + (-0.4140393)x[41,+>
                         + (-0.2760262)x[42,+>

[28b,+> = [(121),2:6,0> =  (-0.8964215)x[37,+>
                         + ( 0.2988072)x[38,+>
                         + (-0.2999998)x[43,+>
                         + (-0.1309310)x[44,+>

[29a,+> = [(211),3:6,0> =  ( 0.5163979)x[39,+>
                         + (-0.5163978)x[40,+>
                         + (-0.4830460)x[41,+>
                         + ( 0.4830455)x[42,+>

[29b,+> = [(121),3:6,0> =  ( 0.3999999)x[43,+>
                         + (-0.9165152)x[44,+>

 [30,+> = [(321),1:6,0> =  ( 0.4330127)x[37,+>
                         + ( 0.4330125)x[38,+>
                         + (-0.7245688)x[43,+>
                         + (-0.3162280)x[44,+>

[31a,+> = [(211),1:6,1> =  ( 1.0000000)x[47,+>

[31b,+> = [(121),1:6,1> =  (-0.2499999)x[45,+>
                         + ( 0.7499998)x[46,+>
                         + (-0.4183299)x[49,+>
                         + (-0.4472140)x[50,+>

[32a,+> = [(211),2:6,1> =  ( 1.0000000)x[48,+>

[32b,+> = [(121),2:6,1> =  (-0.7499998)x[45,+>
                         + ( 0.2900004)x[46,+>
                         + ( 0.4183297)x[49,+>
                         + ( 0.4472140)x[50,+>

 [33,+> = [(321),1:6,1> =  ( 0.6123724)x[45,+>
                         + ( 0.6123723)x[46,+>
                         + ( 0.3415650)x[49,+>
                         + ( 0.3651486)x[50,+>

 [34,+> = [(321),2:6,1> =  ( 0.7302965)x[49,+>
                         + (-0.6831303)x[50,+>

 [35,+> = [(111),1:6,2> =  ( 0.9863020)x[51,+>
                         + ( 0.4797016)x[52,+>
                         + ( 0.1305583)x[57,+>
                         + ( 0.3916747)x[58,+>
                         + ( 0.1092329)x[59,+>
                         + ( 0.3276986)x[60,+>
                         + ( 0.0986927)x[61,+>
                         + ( 0.2960781)x[62,+>
                         + ( 0.0624188)x[63,+>
                         + ( 0.1872566)x[64,+>

 [36,+> = [(111),2:6,2> =  ( 0.3371000)x[52,+>
                         + ( 0.4954336)x[57,+>
                         + (-0.3302891)x[58,+>
                         + ( 0.4149094)x[59,+>
                         + (-0.2763398)x[60,+>
                         + ( 0.3745126)x[61,+>
                         + (-0.2496749)x[62,+>
                         + ( 0.2368630)x[63,+>
                         + (-0.1579086)x[64,+>
```

Table 10.1. Continued

```
[37,+> = [(111),3:6,2> =  ( 0.4654746)x[57,+>
                        + ( 0.4654739)x[58,+>
                        + (-0.0599141)x[59,+>
                        + (-0.0599140)x[60,+>
                        + (-0.2571324)x[61,+>
                        + (-0.2571324)x[62,+>
                        + (-0.4621978)x[63,+>
                        + (-0.4621980)x[64,+>

[38,+> = [(111),4:6,2> =  ( 0.3508236)x[59,+>
                        + ( 0.3508233)x[60,+>
                        + (-0.5547000)x[61,+>
                        + (-0.5547003)x[62,+>
                        + ( 0.2631168)x[63,+>
                        + ( 0.2631174)x[64,+>

[39a,+> = [(211),1:6,2> =  ( 0.8100926)x[51,+>
                        + (-0.3471825)x[52,+>
                        + (-0.0944911)x[57,+>
                        + (-0.2834733)x[58,+>
                        + (-0.0790569)x[59,+>
                        + (-0.2371708)x[60,+>
                        + (-0.0714285)x[61,+>
                        + (-0.2142856)x[62,+>
                        + (-0.0451754)x[63,+>
                        + (-0.1355263)x[64,+>

[39b,+> = [(121),1:6,2> =  ( 0.0731925)x[53,+>
                        + ( 0.6587321)x[54,+>
                        + ( 0.0597615)x[55,+>
                        + ( 0.5378534)x[56,+>
                        + ( 0.3674232)x[65,+>
                        + ( 0.2405353)x[66,+>
                        + ( 0.2405353)x[67,+>
                        + ( 0.1309311)x[68,+>

[40a,+> = [(211),2:6,2> =  ( 0.7319251)x[52,+>
                        + (-0.3585686)x[57,+>
                        + (-0.2390457)x[58,+>
                        + (-0.2999999)x[59,+>
                        + (-0.2000000)x[60,+>
                        + (-0.2710522)x[61,+>
                        + (-0.1807015)x[62,+>
                        + (-0.1714288)x[63,+>
                        + (-0.1142859)x[64,+>

[40b,+> = [(121),2:6,2> =  (-0.6943648)x[53,+>
                        + ( 0.2314549)x[54,+>
                        + (-0.5669470)x[55,+>
                        + ( 0.1889824)x[56,+>
                        + (-0.2323788)x[65,+>
                        + (-0.1521278)x[66,+>
                        + (-0.1521279)x[67,+>
                        + (-0.0828081)x[68,+>

[41a,+> = [(211),3:6,2> =  ( 0.6191392)x[57,+>
                        + (-0.3499481)x[58,+>
                        + (-0.4954867)x[59,+>
                        + ( 0.0450438)x[60,+>
                        + (-0.2950391)x[61,+>
                        + ( 0.1933145)x[62,+>
                        + ( 0.0386096)x[63,+>
                        + ( 0.3474850)x[64,+>

[41b,+> = [(121),3:6,2> =  ( 0.1042574)x[53,+>
                        + ( 0.3127720)x[54,+>
                        + (-0.1276882)x[55,+>
                        + (-0.3830647)x[56,+>
                        + ( 0.4884759)x[65,+>
                        + (-0.5367773)x[66,+>
                        + ( 0.0342626)x[67,+>
                        + (-0.4476033)x[68,+>
```

Table 10.1. Continued

```
[42a,+> = [(211),4:6,2> =  ( 0.5107542)x[58,+>
                         + (-0.2848844)x[59,+>
                         + (-0.5697696)x[60,+>
                         + ( 0.0321747)x[61,+>
                         + (-0.2252215)x[62,+>
                         + ( 0.4476766)x[63,+>
                         + ( 0.2848848)x[64,+>

[42b,+> = [(121),4:6,2> =  (-0.3077110)x[53,+>
                         + ( 0.0879174)x[54,+>
                         + ( 0.3768667)x[55,+>
                         + (-0.1076763)x[56,+>
                         + (-0.2574493)x[65,+>
                         + (-0.4093126)x[66,+>
                         + ( 0.6741605)x[67,+>
                         + ( 0.2359075)x[68,+>

[43a,+> = [(211),5:6,2> =  ( 0.5163985)x[59,+>
                         + (-0.5163977)x[60,+>
                         + (-0.4082476)x[61,+>
                         + ( 0.4082481)x[62,+>
                         + (-0.2581986)x[63,+>
                         + ( 0.2581993)x[64,+>

[43b,+> = [(121),5:6,2> =  (-0.2788865)x[53,+>
                         + (-0.2788865)x[54,+>
                         + ( 0.3415652)x[55,+>
                         + ( 0.3415654)x[56,+>
                         + ( 0.0666653)x[65,+>
                         + ( 0.1527527)x[66,+>
                         + ( 0.1527529)x[67,+>
                         + (-0.7483314)x[68,+>

[44a,+> = [(211),6:6,2> =  ( 0.3779666)x[61,+>
                         + (-0.3779644)x[62,+>
                         + (-0.5976132)x[63,+>
                         + ( 0.5976140)x[64,+>

[44b,+> = [(121),6:6,2> =  (-0.2581992)x[53,+>
                         + (-0.2581992)x[54,+>
                         + ( 0.3162273)x[55,+>
                         + ( 0.3162281)x[56,+>
                         + ( 0.4320500)x[65,+>
                         + (-0.4242641)x[66,+>
                         + (-0.4242638)x[67,+>
                         + ( 0.3464091)x[68,+>

 [45,+> = [(321),1:6,2> =  ( 0.5123475)x[53,+>
                         + (-0.0731923)x[54,+>
                         + (-0.1792844)x[55,+>
                         + ( 0.5378529)x[56,+>
                         + (-0.3674230)x[65,+>
                         + (-0.5077964)x[66,+>
                         + ( 0.0267261)x[67,+>
                         + (-0.1309311)x[68,+>

 [46,+> = [(321),2:6,2> =  ( 0.5070925)x[54,+>
                         + ( 0.5175495)x[55,+>
                         + (-0.1035097)x[56,+>
                         + (-0.4242636)x[65,+>
                         + (-0.0462910)x[66,+>
                         + (-0.5092010)x[67,+>
                         + (-0.1511862)x[68,+>

 [47,+> = [(111),1:6,3> =  ( 0.4225777)x[71,+>
                         + (-0.4225760)x[72,+>
                         + (-0.5669466)x[73,+>
                         + ( 0.5669470)x[74,+>

[48a,+> = [(211),1:6,3> =  ( 0.9063269)x[71,+>
                         + ( 0.1970271)x[72,+>
                         + ( 0.2643403)x[73,+>
                         + (-0.2643405)x[74,+>
```

Table 10.1. Continued

```
[48b,+> = [(121),1:6,3> =  (-0.2211630)x[69,+>
                         + ( 0.6634887)x[70,+>
                         + (-0.6264113)x[75,+>
                         + ( 0.0329690)x[76,+>
                         + (-0.3426246)x[77,+>

[49a,+> = [(211),2:6,3> =  ( 0.8846518)x[72,+>
                         + (-0.3296902)x[73,+>
                         + ( 0.3296902)x[74,+>

[49b,+> = [(121),2:6,3> =  (-0.6304884)x[69,+>
                         + ( 0.0788112)x[70,+>
                         + ( 0.1057360)x[75,+>
                         + ( 0.6344164)x[76,+>
                         + ( 0.4273276)x[77,+>

[50a,+> = [(211),3:6,3> =  ( 0.7071067)x[73,+>
                         + ( 0.7071068)x[74,+>

[50b,+> = [(121),3:6,3> =  (-0.4225772)x[69,+>
                         + ( 0.4225769)x[70,+>
                         + ( 0.5669466)x[75,+>
                         + (-0.5669470)x[76,+>

 [51,+> = [(321),1:6,3> =  ( 0.6123723)x[69,+>
                         + ( 0.6123725)x[70,+>
                         + ( 0.2738613)x[75,+>
                         + ( 0.2738614)x[76,+>
                         + ( 0.3162278)x[77,+>

 [52,+> = [(321),2:6,3> =  ( 0.4472134)x[75,+>
                         + ( 0.4472132)x[76,+>
                         + (-0.7745970)x[77,+>

 [53,+> = [(111),1:6,4> =  ( 0.5863023)x[78,+>
                         + ( 0.4797021)x[79,+>
                         + ( 0.1020621)x[84,+>
                         + ( 0.3061863)x[85,+>
                         + ( 0.1636631)x[86,+>
                         + ( 0.4909895)x[87,+>
                         + ( 0.0735613)x[88,+>
                         + ( 0.2206838)x[89,+>

 [54,+> = [(111),2:6,4> =  ( 0.3370996)x[79,+>
                         + ( 0.3872988)x[84,+>
                         + (-0.2581993)x[85,+>
                         + ( 0.6210584)x[86,+>
                         + (-0.4140394)x[87,+>
                         + ( 0.2791458)x[88,+>
                         + (-0.1860970)x[89,+>

 [55,+> = [(111),3:6,4> =  ( 0.3162274)x[84,+>
                         + ( 0.3162274)x[85,+>
                         + ( 0.0845141)x[86,+>
                         + ( 0.0845146)x[87,+>
                         + (-0.6267825)x[88,+>
                         + (-0.6267844)x[89,+>

[56a,+> = [(211),1:6,4> =  ( 0.8100928)x[78,+>
                         + (-0.3471828)x[79,+>
                         + (-0.0738671)x[84,+>
                         + (-0.2216013)x[85,+>
                         + (-0.1184506)x[86,+>
                         + (-0.3553520)x[87,+>
                         + (-0.0532397)x[88,+>
                         + (-0.1597191)x[89,+>

[56b,+> = [(121),1:6,4> =  ( 0.0852944)x[80,+>
                         + ( 0.7676496)x[81,+>
                         + ( 0.0406624)x[82,+>
                         + ( 0.3659621)x[83,+>
                         + ( 0.3433033)x[90,+>
                         + ( 0.3433033)x[91,+>
                         + ( 0.1792846)x[92,+>
```

Table 10.1. Continued

```
[57a,+> = [(211),2:6,4> =  ( 0.7319250)x[79,+>
                         + (-0.2803062)x[84,+>
                         + (-0.1868710)x[85,+>
                         + (-0.4494890)x[86,+>
                         + (-0.2996597)x[87,+>
                         + (-0.2020308)x[88,+>
                         + (-0.1346873)x[89,+>

[57b,+> = [(121),2:6,4> =  (-0.8091741)x[80,+>
                         + ( 0.2697246)x[81,+>
                         + (-0.3857576)x[82,+>
                         + ( 0.1285860)x[83,+>
                         + (-0.2171240)x[90,+>
                         + (-0.2171240)x[91,+>
                         + (-0.1133896)x[92,+>

[58a,+> = [(211),3:6,4> =  ( 0.8096638)x[84,+>
                         + (-0.1235082)x[85,+>
                         + (-0.5171394)x[86,+>
                         + (-0.0330087)x[87,+>
                         + ( 0.0271996)x[88,+>
                         + ( 0.2448014)x[89,+>

[58b,+> = [(121),3:6,4> =  ( 0.0990385)x[80,+>
                         + ( 0.2971144)x[81,+>
                         + (-0.2077438)x[82,+>
                         + (-0.6232314)x[83,+>
                         + (-0.3029522)x[90,+>
                         + ( 0.4942888)x[91,+>
                         + (-0.3663852)x[92,+>

[59a,+> = [(211),4:6,4> =  ( 0.8001882)x[85,+>
                         + (-0.1132195)x[86,+>
                         + (-0.5283582)x[87,+>
                         + ( 0.2518982)x[88,+>
                         + ( 0.0653066)x[89,+>

[59b,+> = [(121),4:6,4> =  (-0.2853453)x[80,+>
                         + (-0.0543516)x[81,+>
                         + ( 0.5985467)x[82,+>
                         + ( 0.1140091)x[83,+>
                         + (-0.5469033)x[90,+>
                         + ( 0.3828321)x[91,+>
                         + ( 0.3141719)x[92,+>

[60a,+> = [(211),5:6,4> =  ( 0.2898888)x[86,+>
                         + (-0.2898812)x[87,+>
                         + (-0.6449558)x[88,+>
                         + ( 0.6449537)x[89,+>

[60b,+> = [(121),5:6,4> =  (-0.1565572)x[80,+>
                         + (-0.1565564)x[81,+>
                         + ( 0.3283953)x[82,+>
                         + ( 0.3283945)x[83,+>
                         + ( 0.2100434)x[90,+>
                         + ( 0.2100429)x[91,+>
                         + (-0.8043987)x[92,+>

 [61,+> = [(321),1:6,4> =  ( 0.4714043)x[80,+>
                         + ( 0.1767750)x[81,+>
                         + (-0.1545039)x[82,+>
                         + ( 0.4635113)x[83,+>
                         + (-0.6324565)x[90,+>
                         + (-0.2371719)x[91,+>
                         + (-0.2270740)x[92,+>

 [62,+> = [(321),2:6,4> =  ( 0.4370047)x[81,+>
                         + ( 0.5624993)x[82,+>
                         + (-0.3541661)x[83,+>
                         + (-0.5863019)x[91,+>
                         + (-0.1530943)x[92,+>
```

Table 10.1. Continued

```
[63a,+> = [(211),1:6,5> =  ( 1.0000000)x[95,+>

[63b,+> = [(121),1:6,5> =  (-0.2499995)x[93,+>
                         + ( 0.7499989)x[94,+>
                         + (-0.6123739)x[97,+>

[64a,+> = [(211),2:6,5> =  ( 1.0000000)x[96,+>

[64b,+> = [(121),2:6,5> =  (-0.7499998)x[93,+>
                         + ( 0.2500017)x[94,+>
                         + ( 0.6123719)x[97,+>

 [65,+> = [(321),1:6,5> =  ( 0.6123723)x[93,+>
                         + ( 0.6123726)x[94,+>
                         + ( 0.5000000)x[97,+>

 [66,+> = [(111),1:6,6> =  ( 0.5863014)x[98,+>
                         + ( 0.4797011)x[99,+>
                         + ( 0.2064311)x[102,+>
                         + ( 0.6192929)x[103,+>

 [67,+> = [(111),2:6,6> =  ( 0.3371005)x[99,+>
                         + ( 0.7833493)x[102,+>
                         + (-0.5222328)x[103,+>

[68a,+> = [(211),1:6,6> =  ( 0.8100922)x[98,+>
                         + (-0.3471823)x[99,+>
                         + (-0.1494039)x[102,+>
                         + (-0.4482114)x[103,+>

[68b,+> = [(121),1:6,6> =  ( 0.0944915)x[100,+>
                         + ( 0.8504195)x[101,+>
                         + ( 0.5175499)x[104,+>

[69a,+> = [(211),2:6,6> =  ( 0.7319251)x[99,+>
                         + (-0.5669469)x[102,+>
                         + (-0.3779641)x[102,+>

[69b,+> = [(121),2:6,6> =  (-0.8764220)x[100,+>
                         + ( 0.2988072)x[101,+>
                         + (-0.3273255)x[104,+>

 [70,+> = [(321),1:6,6> =  ( 0.4330140)x[100,+>
                         + ( 0.4330116)x[101,+>
                         + (-0.7905692)x[104,+>

 [1a,-> = [(211),1:1,1> =  ( 1.0000000)x[1,->

 [1b,-> = [(121),1:1,1> =  ( 1.0000000)x[2,->

 [2,-> = [(111),1:3,1> =  ( 0.4999999)x[4,->
                        + (-0.6454973)x[6,->
                        + (-0.5773501)x[8,->

 [3a,-> = [(211),1:3,1> =  ( 0.8660254)x[4,->
                         + ( 0.3726780)x[6,->
                         + ( 0.3333333)x[8,->

 [3b,-> = [(121),1:3,1> =  ( 0.8660254)x[3,->
                         + ( 0.3726780)x[5,->
                         + ( 0.3333333)x[7,->

 [4a,-> = [(211),2:3,1> =  ( 0.6666666)x[6,->
                         + (-0.7453560)x[8,->

 [4b,-> = [(121),2:3,1> =  ( 0.6666666)x[5,->
                         + (-0.7453560)x[7,->

 [5,-> = [(321),1:3,1> =  ( 0.4999999)x[3,->
                        + (-0.6454973)x[5,->
                        + (-0.5773502)x[7,->
```

Table 10.1. Continued

```
[6a,-> = [(211),1:3,2> =  ( 1.0000000)x[10,->

[6b,-> = [(121),1:3,2> =  (-1.0000000)x[9,->

 [7,-> = [(111),1:3,3> =  ( 0.5000001)x[12,->
                        + (-0.8660253)x[14,->

[8a,-> = [(211),1:3,3> =  ( 0.8660254)x[12,->
                        + ( 0.4999999)x[14,->

[8b,-> = [(121),1:3,3> =  ( 0.8660256)x[11,->
                        + ( 0.4999996)x[13,->

 [9,-> = [(321),1:3,3> =  ( 0.5000001)x[11,->
                        + (-0.8660253)x[13,->

[10,-> = [(111),1:5,1> =  ( 0.5590170)x[16,->
                        + (-0.5123475)x[17,->
                        + (-0.5477225)x[19,->
                        + (-0.2415229)x[22,->
                        + (-0.2160246)x[24,->
                        + (-0.1414215)x[26,->

[11,-> = [(111),2:5,1> =  ( 0.3651486)x[17,->
                        + (-0.3415647)x[19,->
                        + (-0.2581988)x[22,->
                        + ( 0.6350852)x[24,->
                        + (-0.5291504)x[26,->

[12a,-> = [(211),1:5,1> =  ( 0.8291562)x[16,->
                        + ( 0.3454246)x[17,->
                        + ( 0.3692745)x[19,->
                        + ( 0.1628347)x[22,->
                        + ( 0.1456438)x[24,->
                        + ( 0.0953464)x[26,->

[12b,-> = [(121),1:5,1> =  ( 0.6784005)x[15,->
                        + ( 0.1151415)x[18,->
                        + ( 0.1230915)x[20,->
                        + ( 0.4885042)x[21,->
                        + ( 0.4369312)x[23,->
                        + ( 0.2860391)x[25,->

[13a,-> = [(211),2:5,1> =  ( 0.6963108)x[17,->
                        + (-0.4070868)x[19,->
                        + (-0.1230914)x[22,->
                        + (-0.5642438)x[24,->
                        + ( 0.1261312)x[26,->

[13b,-> = [(121),2:5,1> =  (-0.1994312)x[15,->
                        + ( 0.3916745)x[18,->
                        + (-0.7327561)x[20,->
                        + ( 0.3077285)x[21,->
                        + (-0.1789067)x[23,->
                        + ( 0.3783942)x[25,->

[14a,-> = [(211),3:5,1> =  ( 0.5303300)x[19,->
                        + (-0.6236094)x[22,->
                        + (-0.3486082)x[24,->
                        + (-0.4564359)x[26,->

[14b,-> = [(121),3:5,1> =  (-0.4330125)x[15,->
                        + (-0.6614380)x[18,->
                        + (-0.1767768)x[20,->
                        + ( 0.3118047)x[21,->
                        + ( 0.4880517)x[23,->
                        + ( 0.0912871)x[25,->

[15a,-> = [(211),4:5,1> =  ( 0.6666669)x[22,->
                        + (-0.2981420)x[24,->
                        + (-0.6831300)x[26,->
```

Table 10.1. Continued

```
[15b,->  = [(121),4:5,1> =  ( 0.6666668)x[21,->
                          + (-0.2981421)x[23,->
                          + (-0.6831300)x[25,->

 [16,->  = [(321),1:5,1> =  ( 0.5590170)x[15,->
                          + (-0.5123475)x[18,->
                          + (-0.5477225)x[20,->
                          + (-0.2415230)x[21,->
                          + (-0.2160246)x[23,->
                          + (-0.1414216)x[25,->

 [17,->  = [(321),2:5,1> =  ( 0.3651485)x[18,->
                          + (-0.3415647)x[20,->
                          + (-0.2581989)x[21,->
                          + ( 0.6350850)x[23,->
                          + (-0.5291508)x[25,->

 [18,->  = [(111),1:5,2> =  ( 0.5000000)x[27,->
                          + (-0.5916080)x[30,->
                          + (-0.6324555)x[32,->

[19a,->  = [(211),1:5,2> =  ( 0.8660254)x[27,->
                          + ( 0.3415650)x[30,->
                          + ( 0.3651484)x[32,->

[19b,->  = [(121),1:5,2> =  (-0.8660253)x[28,->
                          + (-0.3415650)x[29,->
                          + (-0.3651487)x[31,->

[20a,->  = [(211),2:5,2> =  ( 0.7302966)x[30,->
                          + (-0.6831302)x[32,->

[20b,->  = [(121),2:5,2> =  (-0.7302962)x[29,->
                          + ( 0.6831306)x[31,->

 [21,->  = [(321),1:5,2> =  ( 0.5000000)x[28,->
                          + (-0.5916078)x[29,->
                          + (-0.6324557)x[31,->

 [22,->  = [(111),1:5,3> =  ( 0.5590169)x[34,->
                          + (-0.6587322)x[35,->
                          + (-0.3585693)x[37,->
                          + (-0.1936490)x[40,->
                          + (-0.2598074)x[42,->
                          + (-0.1414213)x[44,->

 [23,->  = [(111),2:5,3> =  ( 0.2390469)x[35,->
                          + (-0.4391558)x[37,->
                          + (-0.6324546)x[40,->
                          + ( 0.5656855)x[42,->
                          + (-0.1732045)x[44,->

[24a,->  = [(211),1:5,3> =  ( 0.8291560)x[34,->
                          + ( 0.4441172)x[35,->
                          + ( 0.2417474)x[37,->
                          + ( 0.1305581)x[40,->
                          + ( 0.1751621)x[42,->
                          + ( 0.0953462)x[44,->

[24b,->  = [(121),1:5,3> =  ( 0.6784006)x[33,->
                          + ( 0.1480390)x[36,->
                          + ( 0.0805823)x[38,->
                          + ( 0.3916749)x[39,->
                          + ( 0.5254864)x[41,->
                          + ( 0.2860393)x[43,->

[25a,->  = [(211),2:5,3> =  ( 0.5582910)x[35,->
                          + (-0.4273526)x[37,->
                          + (-0.0615457)x[40,->
                          + (-0.6881016)x[42,->
                          + (-0.1685500)x[44,->
```

Table 10.1. Continued

```
[25b,->  = [(121),2:5,3>  =  (-0.3198013)x[33,->
                           + (-0.0697855)x[36,->
                           + (-0.7692342)x[38,->
                           + ( 0.4923650)x[39,->
                           + ( 0.0550483)x[41,->
                           + ( 0.2359706)x[43,->

[26a,->  = [(211),3:5,3>  =  ( 0.6614376)x[37,->
                           + (-0.6123725)x[40,->
                           + (-0.2738615)x[42,->
                           + (-0.3354106)x[44,->

[26b,->  = [(121),3:5,3>  =  (-0.3535534)x[33,->
                           + (-0.6943651)x[36,->
                           + ( 0.2834732)x[38,->
                           + ( 0.5477223)x[41,->
                           + ( 0.1118041)x[43,->

[27a,->  = [(211),4:5,3>  =  ( 0.4082483)x[40,->
                           + ( 0.1825746)x[42,->
                           + (-0.8944271)x[44,->

[27b,->  = [(121),4:5,3>  =  ( 0.4082484)x[39,->
                           + ( 0.1825752)x[41,->
                           + (-0.8944270)x[43,->

 [28,->  = [(321),1:5,3>  =  ( 0.5590171)x[33,->
                           + (-0.6587324)x[36,->
                           + (-0.3585688)x[38,->
                           + (-0.1936493)x[39,->
                           + (-0.2598075)x[41,->
                           + (-0.1414217)x[43,->

 [29,->  = [(321),2:5,3>  =  ( 0.2390466)x[36,->
                           + (-0.4391550)x[38,->
                           + (-0.6324562)x[39,->
                           + ( 0.5656843)x[41,->
                           + (-0.1732049)x[43,->

 [30,->  = [(111),1:5,4>  =  ( 0.5000002)x[45,->
                           + (-0.8660253)x[48,->

[31a,->  = [(211),1:5,4>  =  ( 0.8660255)x[45,->
                           + ( 0.4999999)x[48,->

[31b,->  = [(121),1:5,4>  =  (-0.8660255)x[46,->
                           + (-0.4999997)x[47,->

 [32,->  = [(321),1:5,4>  =  ( 0.5000002)x[46,->
                           + (-0.8660253)x[47,->

 [33,->  = [(111),1:5,5>  =  ( 0.5590173)x[50,->
                           + (-0.7499999)x[51,->
                           + (-0.3535532)x[54,->

[34a,->  = [(211),1:5,5>  =  ( 0.8291563)x[50,->
                           + ( 0.5056497)x[51,->
                           + ( 0.2383654)x[54,->

[34b,->  = [(121),1:5,5>  =  ( 0.6784017)x[49,->
                           + ( 0.1685497)x[52,->
                           + ( 0.7150959)x[53,->

[35a,->  = [(211),2:5,5>  =  ( 0.4264016)x[51,->
                           + (-0.9045340)x[54,->

[35b,->  = [(121),2:5,5>  =  (-0.4767325)x[49,->
                           + (-0.6395994)x[52,->
                           + ( 0.6030247)x[53,->

 [36,->  = [(321),1:5,5>  =  ( 0.5590175)x[49,->
                           + (-0.7499995)x[52,->
                           + (-0.3535537)x[53,->
```

B. Table of the N and Δ 6ω Basis Functions

Table 10.2 lists the N J^{π} and Δ J^{π} 6ω bases. The notation used is

$$|I\rangle = |([m,\pi\rangle,|S\rangle)J\rangle|\tau\rangle \quad ,$$

where $|I\rangle$ are the basis functions and the label I is an index for one set of J, π, and τ. The states $|m,\pi\rangle$ are the space S_3 irreps of Table 10.1, where π is the parity. Table 10.1 gives the total orbital angular momentum of the $|m,\pi\rangle$. The states $|S\rangle$ are the spin S_3 irreps of Table 3.2 and $|\tau\rangle$ are the isospin S_3 irreps of Table 3.1. The total spin and isospin are given as S and τ, respectively, where $|1/2\rangle$ is the [2,1] irrep and $|3/2\rangle$ is the [3] irrep for both $|S\rangle$ and $|\tau\rangle$. The total angular momentum is J, where $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$. This is a shorthand notation, but the coupling rule of Table 3.3 which has been used to give the overall symmetric basis function may be inferred once the irrep label of $|m,\pi\rangle$ is found from Table 10.1.

Table 10.2. The N and Δ 6ω basis functions

N 1/2+

|0 1> = |0 (|0 1,+>,|1/2>) 1/2> |1/2>
|0 2> = |0 (|0 2,+>,|1/2>) 1/2> |1/2>
|0 3> = |0 (|0 7,+>,|1/2>) 1/2> |1/2>
|0 4> = |0 (|0 8,+>,|1/2>) 1/2> |1/2>
|0 5> = |0 (|24,+>,|1/2>) 1/2> |1/2>
|0 6> = |0 (|25,+>,|1/2>) 1/2> |1/2>
|0 7> = |0 (|26,+>,|1/2>) 1/2> |1/2>
|0 8> = |0 (|0 3,+>,|1/2>) 1/2> |1/2>
|0 9> = |0 (|0 6,+>,|3/2>) 1/2> |1/2>
|10> = |0 (|0 9,+>,|1/2>) 1/2> |1/2>
|11> = |0 (|10,+>,|1/2>) 1/2> |1/2>
|12> = |0 (|11,+>,|1/2>) 1/2> |1/2>
|13> = |0 (|11,+>,|3/2>) 1/2> |1/2>
|14> = |0 (|15,+>,|3/2>) 1/2> |1/2>
|15> = |0 (|16,+>,|3/2>) 1/2> |1/2>
|16> = |0 (|17,+>,|3/2>) 1/2> |1/2>
|17> = |0 (|27,+>,|1/2>) 1/2> |1/2>
|18> = |0 (|28,+>,|1/2>) 1/2> |1/2>
|19> = |0 (|29,+>,|1/2>) 1/2> |1/2>
|20> = |0 (|31,+>,|1/2>) 1/2> |1/2>
|21> = |0 (|31,+>,|3/2>) 1/2> |1/2>
|22> = |0 (|32,+>,|1/2>) 1/2> |1/2>
|23> = |0 (|32,+>,|3/2>) 1/2> |1/2>
|24> = |0 (|39,+>,|3/2>) 1/2> |1/2>
|25> = |0 (|40,+>,|3/2>) 1/2> |1/2>
|26> = |0 (|41,+>,|3/2>) 1/2> |1/2>
|27> = |0 (|42,+>,|3/2>) 1/2> |1/2>
|28> = |0 (|43,+>,|3/2>) 1/2> |1/2>
|29> = |0 (|44,+>,|3/2>) 1/2> |1/2>
|30> = |0 (|0 4,+>,|1/2>) 1/2> |1/2>
|31> = |0 (|12,+>,|1/2>) 1/2> |1/2>
|32> = |0 (|30,+>,|1/2>) 1/2> |1/2>
|33> = |0 (|33,+>,|1/2>) 1/2> |1/2>
|34> = |0 (|34,+>,|1/2>) 1/2> |1/2>

N 1/2-

|0 1> = |0 (|0 2,->,|1/2>) 1/2> |1/2>
|0 2> = |0 (|10,->,|1/2>) 1/2> |1/2>
|0 3> = |0 (|11,->,|1/2>) 1/2> |1/2>
|0 4> = |0 (|0 1,->,|1/2>) 1/2> |1/2>
|0 5> = |0 (|0 1,->,|3/2>) 1/2> |1/2>
|0 6> = |0 (|0 3,->,|1/2>) 1/2> |1/2>
|0 7> = |0 (|0 3,->,|3/2>) 1/2> |1/2>
|0 8> = |0 (|0 4,->,|1/2>) 1/2> |1/2>
|0 9> = |0 (|0 4,->,|3/2>) 1/2> |1/2>
|10> = |0 (|0 6,->,|3/2>) 1/2> |1/2>
|11> = |0 (|12,->,|1/2>) 1/2> |1/2>
|12> = |0 (|12,->,|3/2>) 1/2> |1/2>
|13> = |0 (|13,->,|1/2>) 1/2> |1/2>
|14> = |0 (|13,->,|3/2>) 1/2> |1/2>
|15> = |0 (|14,->,|1/2>) 1/2> |1/2>
|16> = |0 (|14,->,|3/2>) 1/2> |1/2>
|17> = |0 (|15,->,|1/2>) 1/2> |1/2>
|18> = |0 (|15,->,|3/2>) 1/2> |1/2>
|19> = |0 (|19,->,|3/2>) 1/2> |1/2>
|20> = |0 (|20,->,|3/2>) 1/2> |1/2>
|21> = |0 (|0 5,->,|1/2>) 1/2> |1/2>
|22> = |0 (|16,->,|1/2>) 1/2> |1/2>
|23> = |0 (|17,->,|1/2>) 1/2> |1/2>

N 3/2+

|0 1> = |0 (|0 5,+>,|1/2>) 3/2> |1/2>
|0 2> = |0 (|13,+>,|1/2>) 3/2> |1/2>
|0 3> = |0 (|14,+>,|1/2>) 3/2> |1/2>
|0 4> = |0 (|35,+>,|1/2>) 3/2> |1/2>
|0 5> = |0 (|36,+>,|1/2>) 3/2> |1/2>
|0 6> = |0 (|37,+>,|1/2>) 3/2> |1/2>
|0 7> = |0 (|38,+>,|1/2>) 3/2> |1/2>
|0 8> = |0 (|0 3,+>,|3/2>) 3/2> |1/2>
|0 9> = |0 (|0 6,+>,|1/2>) 3/2> |1/2>
|10> = |0 (|0 6,+>,|3/2>) 3/2> |1/2>
|11> = |0 (|0 9,+>,|3/2>) 3/2> |1/2>
|12> = |0 (|10,+>,|3/2>) 3/2> |1/2>
|13> = |0 (|11,+>,|1/2>) 3/2> |1/2>
|14> = |0 (|11,+>,|3/2>) 3/2> |1/2>
|15> = |0 (|15,+>,|1/2>) 3/2> |1/2>
|16> = |0 (|15,+>,|3/2>) 3/2> |1/2>
|17> = |0 (|16,+>,|1/2>) 3/2> |1/2>
|18> = |0 (|16,+>,|3/2>) 3/2> |1/2>
|19> = |0 (|17,+>,|1/2>) 3/2> |1/2>
|20> = |0 (|17,+>,|3/2>) 3/2> |1/2>
|21> = |0 (|19,+>,|3/2>) 3/2> |1/2>
|22> = |0 (|27,+>,|3/2>) 3/2> |1/2>
|23> = |0 (|28,+>,|3/2>) 3/2> |1/2>
|24> = |0 (|29,+>,|3/2>) 3/2> |1/2>
|25> = |0 (|31,+>,|1/2>) 3/2> |1/2>
|26> = |0 (|31,+>,|3/2>) 3/2> |1/2>
|27> = |0 (|32,+>,|1/2>) 3/2> |1/2>
|28> = |0 (|32,+>,|3/2>) 3/2> |1/2>
|29> = |0 (|39,+>,|1/2>) 3/2> |1/2>
|30> = |0 (|39,+>,|3/2>) 3/2> |1/2>
|31> = |0 (|40,+>,|1/2>) 3/2> |1/2>
|32> = |0 (|40,+>,|3/2>) 3/2> |1/2>
|33> = |0 (|41,+>,|1/2>) 3/2> |1/2>
|34> = |0 (|41,+>,|3/2>) 3/2> |1/2>
|35> = |0 (|42,+>,|1/2>) 3/2> |1/2>
|36> = |0 (|42,+>,|3/2>) 3/2> |1/2>
|37> = |0 (|43,+>,|1/2>) 3/2> |1/2>
|38> = |0 (|43,+>,|3/2>) 3/2> |1/2>
|39> = |0 (|44,+>,|1/2>) 3/2> |1/2>
|40> = |0 (|44,+>,|3/2>) 3/2> |1/2>
|41> = |0 (|48,+>,|3/2>) 3/2> |1/2>
|42> = |0 (|49,+>,|3/2>) 3/2> |1/2>
|43> = |0 (|50,+>,|3/2>) 3/2> |1/2>
|44> = |0 (|0 4,+>,|1/2>) 3/2> |1/2>
|45> = |0 (|12,+>,|1/2>) 3/2> |1/2>
|46> = |0 (|18,+>,|1/2>) 3/2> |1/2>
|47> = |0 (|33,+>,|1/2>) 3/2> |1/2>
|48> = |0 (|34,+>,|1/2>) 3/2> |1/2>
|49> = |0 (|45,+>,|1/2>) 3/2> |1/2>
|50> = |0 (|46,+>,|1/2>) 3/2> |1/2>

Table 10.2. Continued

N 3/2-

[1> = [([2,->,[1/2>) 3/2> [1/2>
[2> = [([10,->,[1/2>) 3/2> [1/2>
[3> = [([11,->,[1/2>) 3/2> [1/2>
[4> = [([18,->,[1/2>) 3/2> [1/2>
[5> = [([1,->,[1/2>) 3/2> [1/2>
[6> = [([1,->,[3/2>) 3/2> [1/2>
[7> = [([3,->,[1/2>) 3/2> [1/2>
[8> = [([3,->,[3/2>) 3/2> [1/2>
[9> = [([4,->,[1/2>) 3/2> [1/2>
[10> = [([4,->,[3/2>) 3/2> [1/2>
[11> = [([6,->,[1/2>) 3/2> [1/2>
[12> = [([6,->,[3/2>) 3/2> [1/2>
[13> = [([8,->,[3/2>) 3/2> [1/2>
[14> = [([12,->,[1/2>) 3/2> [1/2>
[15> = [([12,->,[3/2>) 3/2> [1/2>
[16> = [([13,->,[1/2>) 3/2> [1/2>
[17> = [([13,->,[3/2>) 3/2> [1/2>
[18> = [([14,->,[1/2>) 3/2> [1/2>
[19> = [([14,->,[3/2>) 3/2> [1/2>
[20> = [([15,->,[1/2>) 3/2> [1/2>
[21> = [([15,->,[3/2>) 3/2> [1/2>
[22> = [([19,->,[1/2>) 3/2> [1/2>
[23> = [([19,->,[3/2>) 3/2> [1/2>
[24> = [([20,->,[1/2>) 3/2> [1/2>
[25> = [([20,->,[3/2>) 3/2> [1/2>
[26> = [([24,->,[3/2>) 3/2> [1/2>
[27> = [([25,->,[3/2>) 3/2> [1/2>
[28> = [([26,->,[3/2>) 3/2> [1/2>
[29> = [([27,->,[3/2>) 3/2> [1/2>
[30> = [([5,->,[1/2>) 3/2> [1/2>
[31> = [([16,->,[1/2>) 3/2> [1/2>
[32> = [([17,->,[1/2>) 3/2> [1/2>
[33> = [([21,->,[1/2>) 3/2> [1/2>

N 5/2+

[1> = [([5,+>,[1/2>) 5/2> [1/2>
[2> = [([13,+>,[1/2>) 5/2> [1/2>
[3> = [([14,+>,[1/2>) 5/2> [1/2>
[4> = [([35,+>,[1/2>) 5/2> [1/2>
[5> = [([36,+>,[1/2>) 5/2> [1/2>
[6> = [([37,+>,[1/2>) 5/2> [1/2>
[7> = [([38,+>,[1/2>) 5/2> [1/2>
[8> = [([47,+>,[1/2>) 5/2> [1/2>
[9> = [([6,+>,[1/2>) 5/2> [1/2>
[10> = [([6,+>,[3/2>) 5/2> [1/2>
[11> = [([11,+>,[3/2>) 5/2> [1/2>
[12> = [([15,+>,[1/2>) 5/2> [1/2>
[13> = [([15,+>,[3/2>) 5/2> [1/2>
[14> = [([16,+>,[1/2>) 5/2> [1/2>
[15> = [([16,+>,[3/2>) 5/2> [1/2>
[16> = [([17,+>,[1/2>) 5/2> [1/2>
[17> = [([17,+>,[3/2>) 5/2> [1/2>
[18> = [([19,+>,[1/2>) 5/2> [1/2>
[19> = [([19,+>,[3/2>) 5/2> [1/2>
[20> = [([22,+>,[3/2>) 5/2> [1/2>
[21> = [([23,+>,[3/2>) 5/2> [1/2>
[22> = [([31,+>,[3/2>) 5/2> [1/2>
[23> = [([32,+>,[3/2>) 5/2> [1/2>
[24> = [([39,+>,[1/2>) 5/2> [1/2>
[25> = [([39,+>,[3/2>) 5/2> [1/2>
[26> = [([40,+>,[1/2>) 5/2> [1/2>
[27> = [([40,+>,[3/2>) 5/2> [1/2>
[28> = [([41,+>,[1/2>) 5/2> [1/2>
[29> = [([41,+>,[3/2>) 5/2> [1/2>
[30> = [([42,+>,[1/2>) 5/2> [1/2>
[31> = [([42,+>,[3/2>) 5/2> [1/2>
[32> = [([43,+>,[1/2>) 5/2> [1/2>
[33> = [([43,+>,[3/2>) 5/2> [1/2>
[34> = [([44,+>,[1/2>) 5/2> [1/2>
[35> = [([44,+>,[3/2>) 5/2> [1/2>
[36> = [([48,+>,[1/2>) 5/2> [1/2>
[37> = [([48,+>,[3/2>) 5/2> [1/2>
[38> = [([49,+>,[1/2>) 5/2> [1/2>
[39> = [([49,+>,[3/2>) 5/2> [1/2>
[40> = [([50,+>,[1/2>) 5/2> [1/2>
[41> = [([50,+>,[3/2>) 5/2> [1/2>
[42> = [([56,+>,[3/2>) 5/2> [1/2>
[43> = [([57,+>,[3/2>) 5/2> [1/2>
[44> = [([58,+>,[3/2>) 5/2> [1/2>
[45> = [([59,+>,[3/2>) 5/2> [1/2>
[46> = [([60,+>,[3/2>) 5/2> [1/2>
[47> = [([18,+>,[1/2>) 5/2> [1/2>
[48> = [([20,+>,[1/2>) 5/2> [1/2>
[49> = [([45,+>,[1/2>) 5/2> [1/2>
[50> = [([46,+>,[1/2>) 5/2> [1/2>
[51> = [([51,+>,[1/2>) 5/2> [1/2>
[52> = [([52,+>,[1/2>) 5/2> [1/2>

Table 10.2. Continued

N 5/2-

[1> = [([7,->,[1/2>) 5/2> [1/2>
[2> = [([18,->,[1/2>) 5/2> [1/2>
[3> = [([22,->,[1/2>) 5/2> [1/2>
[4> = [([23,->,[1/2>) 5/2> [1/2>
[5> = [([1,->,[3/2>) 5/2> [1/2>
[6> = [([3,->,[3/2>) 5/2> [1/2>
[7> = [([4,->,[3/2>) 5/2> [1/2>
[8> = [([6,->,[1/2>) 5/2> [1/2>
[9> = [([6,->,[3/2>) 5/2> [1/2>
[10> = [([8,->,[1/2>) 5/2> [1/2>
[11> = [([8,->,[3/2>) 5/2> [1/2>
[12> = [([12,->,[3/2>) 5/2> [1/2>
[13> = [([13,->,[3/2>) 5/2> [1/2>
[14> = [([14,->,[3/2>) 5/2> [1/2>
[15> = [([15,->,[3/2>) 5/2> [1/2>
[16> = [([19,->,[1/2>) 5/2> [1/2>
[17> = [([19,->,[3/2>) 5/2> [1/2>
[18> = [([20,->,[1/2>) 5/2> [1/2>
[19> = [([20,->,[3/2>) 5/2> [1/2>
[20> = [([24,->,[1/2>) 5/2> [1/2>
[21> = [([24,->,[3/2>) 5/2> [1/2>
[22> = [([25,->,[1/2>) 5/2> [1/2>
[23> = [([25,->,[3/2>) 5/2> [1/2>
[24> = [([26,->,[1/2>) 5/2> [1/2>
[25> = [([26,->,[3/2>) 5/2> [1/2>
[26> = [([27,->,[1/2>) 5/2> [1/2>
[27> = [([27,->,[3/2>) 5/2> [1/2>
[28> = [([31,->,[3/2>) 5/2> [1/2>
[29> = [([9,->,[1/2>) 5/2> [1/2>
[30> = [([21,->,[1/2>) 5/2> [1/2>
[31> = [([28,->,[1/2>) 5/2> [1/2>
[32> = [([29,->,[1/2>) 5/2> [1/2>

N 7/2+

[1> = [([21,+>,[1/2>) 7/2> [1/2>
[2> = [([47,+>,[1/2>) 7/2> [1/2>
[3> = [([53,+>,[1/2>) 7/2> [1/2>
[4> = [([54,+>,[1/2>) 7/2> [1/2>
[5> = [([55,+>,[1/2>) 7/2> [1/2>
[6> = [([6,+>,[3/2>) 7/2> [1/2>
[7> = [([15,+>,[3/2>) 7/2> [1/2>
[8> = [([16,+>,[3/2>) 7/2> [1/2>
[9> = [([17,+>,[3/2>) 7/2> [1/2>
[10> = [([19,+>,[1/2>) 7/2> [1/2>
[11> = [([19,+>,[3/2>) 7/2> [1/2>
[12> = [([22,+>,[1/2>) 7/2> [1/2>
[13> = [([22,+>,[3/2>) 7/2> [1/2>
[14> = [([23,+>,[1/2>) 7/2> [1/2>
[15> = [([23,+>,[3/2>) 7/2> [1/2>
[16> = [([39,+>,[3/2>) 7/2> [1/2>
[17> = [([40,+>,[3/2>) 7/2> [1/2>
[18> = [([41,+>,[3/2>) 7/2> [1/2>
[19> = [([42,+>,[3/2>) 7/2> [1/2>
[20> = [([43,+>,[3/2>) 7/2> [1/2>
[21> = [([44,+>,[3/2>) 7/2> [1/2>
[22> = [([48,+>,[1/2>) 7/2> [1/2>
[23> = [([48,+>,[3/2>) 7/2> [1/2>
[24> = [([49,+>,[1/2>) 7/2> [1/2>
[25> = [([49,+>,[3/2>) 7/2> [1/2>
[26> = [([50,+>,[1/2>) 7/2> [1/2>
[27> = [([50,+>,[3/2>) 7/2> [1/2>
[28> = [([56,+>,[1/2>) 7/2> [1/2>
[29> = [([56,+>,[3/2>) 7/2> [1/2>
[30> = [([57,+>,[1/2>) 7/2> [1/2>
[31> = [([57,+>,[3/2>) 7/2> [1/2>
[32> = [([58,+>,[1/2>) 7/2> [1/2>
[33> = [([58,+>,[3/2>) 7/2> [1/2>
[34> = [([59,+>,[1/2>) 7/2> [1/2>
[35> = [([59,+>,[3/2>) 7/2> [1/2>
[36> = [([60,+>,[1/2>) 7/2> [1/2>
[37> = [([60,+>,[3/2>) 7/2> [1/2>
[38> = [([63,+>,[3/2>) 7/2> [1/2>
[39> = [([64,+>,[3/2>) 7/2> [1/2>
[40> = [([20,+>,[1/2>) 7/2> [1/2>
[41> = [([51,+>,[1/2>) 7/2> [1/2>
[42> = [([52,+>,[1/2>) 7/2> [1/2>
[43> = [([61,+>,[1/2>) 7/2> [1/2>
[44> = [([62,+>,[1/2>) 7/2> [1/2>

Table 10.2. Continued

N 7/2-

[1> = [([7,->,[1/2>) 7/2> [1/2>
[2> = [([22,->,[1/2>) 7/2> [1/2>
[3> = [([23,->,[1/2>) 7/2> [1/2>
[4> = [([30,->,[1/2>) 7/2> [1/2>
[5> = [([6,->,[3/2>) 7/2> [1/2>
[6> = [([8,->,[1/2>) 7/2> [1/2>
[7> = [([8,->,[3/2>) 7/2> [1/2>
[8> = [([19,->,[3/2>) 7/2> [1/2>
[9> = [([20,->,[3/2>) 7/2> [1/2>
[10> = [([24,->,[1/2>) 7/2> [1/2>
[11> = [([24,->,[3/2>) 7/2> [1/2>
[12> = [([25,->,[1/2>) 7/2> [1/2>
[13> = [([25,->,[3/2>) 7/2> [1/2>
[14> = [([26,->,[1/2>) 7/2> [1/2>
[15> = [([26,->,[3/2>) 7/2> [1/2>
[16> = [([27,->,[1/2>) 7/2> [1/2>
[17> = [([27,->,[3/2>) 7/2> [1/2>
[18> = [([31,->,[1/2>) 7/2> [1/2>
[19> = [([31,->,[3/2>) 7/2> [1/2>
[20> = [([34,->,[3/2>) 7/2> [1/2>
[21> = [([35,->,[3/2>) 7/2> [1/2>
[22> = [([9,->,[1/2>) 7/2> [1/2>
[23> = [([28,->,[1/2>) 7/2> [1/2>
[24> = [([29,->,[1/2>) 7/2> [1/2>
[25> = [([32,->,[1/2>) 7/2> [1/2>

N 9/2+

[1> = [([21,+>,[1/2>) 9/2> [1/2>
[2> = [([53,+>,[1/2>) 9/2> [1/2>
[3> = [([54,+>,[1/2>) 9/2> [1/2>
[4> = [([55,+>,[1/2>) 9/2> [1/2>
[5> = [([19,+>,[3/2>) 9/2> [1/2>
[6> = [([22,+>,[1/2>) 9/2> [1/2>
[7> = [([22,+>,[3/2>) 9/2> [1/2>
[8> = [([23,+>,[1/2>) 9/2> [1/2>
[9> = [([23,+>,[3/2>) 9/2> [1/2>
[10> = [([48,+>,[3/2>) 9/2> [1/2>
[11> = [([49,+>,[3/2>) 9/2> [1/2>
[12> = [([50,+>,[3/2>) 9/2> [1/2>
[13> = [([56,+>,[1/2>) 9/2> [1/2>
[14> = [([56,+>,[3/2>) 9/2> [1/2>
[15> = [([57,+>,[1/2>) 9/2> [1/2>
[16> = [([57,+>,[3/2>) 9/2> [1/2>
[17> = [([58,+>,[1/2>) 9/2> [1/2>
[18> = [([58,+>,[3/2>) 9/2> [1/2>
[19> = [([59,+>,[1/2>) 9/2> [1/2>
[20> = [([59,+>,[3/2>) 9/2> [1/2>
[21> = [([60,+>,[1/2>) 9/2> [1/2>
[22> = [([60,+>,[3/2>) 9/2> [1/2>
[23> = [([63,+>,[1/2>) 9/2> [1/2>
[24> = [([63,+>,[3/2>) 9/2> [1/2>
[25> = [([64,+>,[1/2>) 9/2> [1/2>
[26> = [([64,+>,[3/2>) 9/2> [1/2>
[27> = [([68,+>,[3/2>) 9/2> [1/2>
[28> = [([69,+>,[3/2>) 9/2> [1/2>
[29> = [([61,+>,[1/2>) 9/2> [1/2>
[30> = [([62,+>,[1/2>) 9/2> [1/2>
[31> = [([65,+>,[1/2>) 9/2> [1/2>

N 9/2-

[1> = [([30,->,[1/2>) 9/2> [1/2>
[2> = [([33,->,[1/2>) 9/2> [1/2>
[3> = [([8,->,[3/2>) 9/2> [1/2>
[4> = [([24,->,[3/2>) 9/2> [1/2>
[5> = [([25,->,[3/2>) 9/2> [1/2>
[6> = [([26,->,[3/2>) 9/2> [1/2>
[7> = [([27,->,[3/2>) 9/2> [1/2>
[8> = [([31,->,[1/2>) 9/2> [1/2>
[9> = [([31,->,[3/2>) 9/2> [1/2>
[10> = [([34,->,[1/2>) 9/2> [1/2>
[11> = [([34,->,[3/2>) 9/2> [1/2>
[12> = [([35,->,[1/2>) 9/2> [1/2>
[13> = [([35,->,[3/2>) 9/2> [1/2>
[14> = [([32,->,[1/2>) 9/2> [1/2>
[15> = [([36,->,[1/2>) 9/2> [1/2>

N 11/2+

[1> = [([66,+>,[1/2>) 11/2> [1/2>
[2> = [([67,+>,[1/2>) 11/2> [1/2>
[3> = [([22,+>,[3/2>) 11/2> [1/2>
[4> = [([23,+>,[3/2>) 11/2> [1/2>
[5> = [([56,+>,[3/2>) 11/2> [1/2>
[6> = [([57,+>,[3/2>) 11/2> [1/2>
[7> = [([58,+>,[3/2>) 11/2> [1/2>
[8> = [([59,+>,[3/2>) 11/2> [1/2>
[9> = [([60,+>,[3/2>) 11/2> [1/2>
[10> = [([63,+>,[1/2>) 11/2> [1/2>
[11> = [([63,+>,[3/2>) 11/2> [1/2>
[12> = [([64,+>,[1/2>) 11/2> [1/2>
[13> = [([64,+>,[3/2>) 11/2> [1/2>
[14> = [([68,+>,[1/2>) 11/2> [1/2>
[15> = [([68,+>,[3/2>) 11/2> [1/2>
[16> = [([69,+>,[1/2>) 11/2> [1/2>
[17> = [([69,+>,[3/2>) 11/2> [1/2>
[18> = [([65,+>,[1/2>) 11/2> [1/2>
[19> = [([70,+>,[1/2>) 11/2> [1/2>

N 11/2-

[1> = [([33,->,[1/2>) 11/2> [1/2>
[2> = [([31,->,[3/2>) 11/2> [1/2>
[3> = [([34,->,[1/2>) 11/2> [1/2>
[4> = [([34,->,[3/2>) 11/2> [1/2>
[5> = [([35,->,[1/2>) 11/2> [1/2>
[6> = [([35,->,[3/2>) 11/2> [1/2>
[7> = [([36,->,[1/2>) 11/2> [1/2>

Table 10.2. Continued

N 13/2+

[1> = [([66,+>,[1/2>) 13/2> [1/2>
[2> = [([67,+>,[1/2>) 13/2> [1/2>
[3> = [([63,+>,[3/2>) 13/2> [1/2>
[4> = [([64,+>,[3/2>) 13/2> [1/2>
[5> = [([68,+>,[1/2>) 13/2> [1/2>
[6> = [([68,+>,[3/2>) 13/2> [1/2>
[7> = [([69,+>,[1/2>) 13/2> [1/2>
[8> = [([69,+>,[3/2>) 13/2> [1/2>
[9> = [([70,+>,[1/2>) 13/2> [1/2>

N 13/2-

[1> = [([34,->,[3/2>) 13/2> [1/2>
[2> = [([35,->,[3/2>) 13/2> [1/2>

N 15/2+

[1> = [([68,+>,[3/2>) 15/2> [1/2>
[2> = [([69,+>,[3/2>) 15/2> [1/2>

Δ 1/2+

[1> = [([5,+>,[3/2>) 1/2> [3/2>
[2> = [([13,+>,[3/2>) 1/2> [3/2>
[3> = [([14,+>,[3/2>) 1/2> [3/2>
[4> = [([35,+>,[3/2>) 1/2> [3/2>
[5> = [([36,+>,[3/2>) 1/2> [3/2>
[6> = [([37,+>,[3/2>) 1/2> [3/2>
[7> = [([38,+>,[3/2>) 1/2> [3/2>
[8> = [([3,+>,[1/2>) 1/2> [3/2>
[9> = [([9,+>,[1/2>) 1/2> [3/2>
[10> = [([10,+>,[1/2>) 1/2> [3/2>
[11> = [([11,+>,[1/2>) 1/2> [3/2>
[12> = [([27,+>,[1/2>) 1/2> [3/2>
[13> = [([28,+>,[1/2>) 1/2> [3/2>
[14> = [([29,+>,[1/2>) 1/2> [3/2>
[15> = [([31,+>,[1/2>) 1/2> [3/2>
[16> = [([31,+>,[1/2>) 1/2> [3/2>

Δ 1/2-

[1> = [([2,->,[3/2>) 1/2> [3/2>
[2> = [([10,->,[3/2>) 1/2> [3/2>
[3> = [([11,->,[3/2>) 1/2> [3/2>
[4> = [([18,->,[3/2>) 1/2> [3/2>
[5> = [([1,->,[1/2>) 1/2> [3/2>
[6> = [([3,->,[1/2>) 1/2> [3/2>
[7> = [([4,->,[1/2>) 1/2> [3/2>
[8> = [([12,->,[1/2>) 1/2> [3/2>
[9> = [([13,->,[1/2>) 1/2> [3/2>
[10> = [([14,->,[1/2>) 1/2> [3/2>
[11> = [([15,->,[1/2>) 1/2> [3/2>

Δ 3/2+

[1> = [([1,+>,[3/2>) 3/2> [3/2>
[2> = [([2,+>,[3/2>) 3/2> [3/2>
[3> = [([5,+>,[3/2>) 3/2> [3/2>
[4> = [([7,+>,[3/2>) 3/2> [3/2>
[5> = [([8,+>,[3/2>) 3/2> [3/2>
[6> = [([13,+>,[3/2>) 3/2> [3/2>
[7> = [([14,+>,[3/2>) 3/2> [3/2>
[8> = [([24,+>,[3/2>) 3/2> [3/2>
[9> = [([25,+>,[3/2>) 3/2> [3/2>
[10> = [([26,+>,[3/2>) 3/2> [3/2>
[11> = [([35,+>,[3/2>) 3/2> [3/2>
[12> = [([36,+>,[3/2>) 3/2> [3/2>
[13> = [([37,+>,[3/2>) 3/2> [3/2>
[14> = [([38,+>,[3/2>) 3/2> [3/2>
[15> = [([47,+>,[3/2>) 3/2> [3/2>
[16> = [([6,+>,[1/2>) 3/2> [3/2>
[17> = [([11,+>,[1/2>) 3/2> [3/2>
[18> = [([15,+>,[1/2>) 3/2> [3/2>
[19> = [([16,+>,[1/2>) 3/2> [3/2>
[20> = [([17,+>,[1/2>) 3/2> [3/2>
[21> = [([31,+>,[1/2>) 3/2> [3/2>
[22> = [([32,+>,[1/2>) 3/2> [3/2>
[23> = [([39,+>,[1/2>) 3/2> [3/2>
[24> = [([40,+>,[1/2>) 3/2> [3/2>
[25> = [([41,+>,[1/2>) 3/2> [3/2>
[26> = [([42,+>,[1/2>) 3/2> [3/2>
[27> = [([43,+>,[1/2>) 3/2> [3/2>
[28> = [([44,+>,[1/2>) 3/2> [3/2>

Δ 3/2-

[1> = [([2,->,[3/2>) 3/2> [3/2>
[2> = [([7,->,[3/2>) 3/2> [3/2>
[3> = [([10,->,[3/2>) 3/2> [3/2>
[4> = [([11,->,[3/2>) 3/2> [3/2>
[5> = [([18,->,[3/2>) 3/2> [3/2>
[6> = [([22,->,[3/2>) 3/2> [3/2>
[7> = [([23,->,[3/2>) 3/2> [3/2>
[8> = [([1,->,[1/2>) 3/2> [3/2>
[9> = [([3,->,[1/2>) 3/2> [3/2>
[10> = [([4,->,[1/2>) 3/2> [3/2>
[11> = [([6,->,[1/2>) 3/2> [3/2>
[12> = [([12,->,[1/2>) 3/2> [3/2>
[13> = [([13,->,[1/2>) 3/2> [3/2>
[14> = [([14,->,[1/2>) 3/2> [3/2>
[15> = [([15,->,[1/2>) 3/2> [3/2>
[16> = [([19,->,[1/2>) 3/2> [3/2>
[17> = [([20,->,[1/2>) 3/2> [3/2>

Table 10.2. Continued

Δ 5/2+

[1> = [([5,+>,[3/2>) 5/2> [3/2>
[2> = [([13,+>,[3/2>) 5/2> [3/2>
[3> = [([14,+>,[3/2>) 5/2> [3/2>
[4> = [([21,+>,[3/2>) 5/2> [3/2>
[5> = [([35,+>,[3/2>) 5/2> [3/2>
[6> = [([36,+>,[3/2>) 5/2> [3/2>
[7> = [([37,+>,[3/2>) 5/2> [3/2>
[8> = [([38,+>,[3/2>) 5/2> [3/2>
[9> = [([47,+>,[3/2>) 5/2> [3/2>
[10> = [([53,+>,[3/2>) 5/2> [3/2>
[11> = [([54,+>,[3/2>) 5/2> [3/2>
[12> = [([55,+>,[3/2>) 5/2> [3/2>
[13> = [([6,+>,[1/2>) 5/2> [3/2>
[14> = [([15,+>,[1/2>) 5/2> [3/2>
[15> = [([16,+>,[1/2>) 5/2> [3/2>
[16> = [([17,+>,[1/2>) 5/2> [3/2>
[17> = [([19,+>,[1/2>) 5/2> [3/2>
[18> = [([39,+>,[1/2>) 5/2> [3/2>
[19> = [([40,+>,[1/2>) 5/2> [3/2>
[20> = [([41,+>,[1/2>) 5/2> [3/2>
[21> = [([42,+>,[1/2>) 5/2> [3/2>
[22> = [([43,+>,[1/2>) 5/2> [3/2>
[23> = [([44,+>,[1/2>) 5/2> [3/2>
[24> = [([48,+>,[1/2>) 5/2> [3/2>
[25> = [([49,+>,[1/2>) 5/2> [3/2>
[26> = [([50,+>,[1/2>) 5/2> [3/2>

Δ 5/2-

[1> = [([2,->,[3/2>) 5/2> [3/2>
[2> = [([7,->,[3/2>) 5/2> [3/2>
[3> = [([10,->,[3/2>) 5/2> [3/2>
[4> = [([11,->,[3/2>) 5/2> [3/2>
[5> = [([18,->,[3/2>) 5/2> [3/2>
[6> = [([22,->,[3/2>) 5/2> [3/2>
[7> = [([23,->,[3/2>) 5/2> [3/2>
[8> = [([30,->,[3/2>) 5/2> [3/2>
[9> = [([6,->,[1/2>) 5/2> [3/2>
[10> = [([8,->,[1/2>) 5/2> [3/2>
[11> = [([19,->,[1/2>) 5/2> [3/2>
[12> = [([20,->,[1/2>) 5/2> [3/2>
[13> = [([24,->,[1/2>) 5/2> [3/2>
[14> = [([25,->,[1/2>) 5/2> [3/2>
[15> = [([26,->,[1/2>) 5/2> [3/2>
[16> = [([27,->,[1/2>) 5/2> [3/2>

Δ 7/2+

[1> = [([5,+>,[3/2>) 7/2> [3/2>
[2> = [([13,+>,[3/2>) 7/2> [3/2>
[3> = [([14,+>,[3/2>) 7/2> [3/2>
[4> = [([21,+>,[3/2>) 7/2> [3/2>
[5> = [([35,+>,[3/2>) 7/2> [3/2>
[6> = [([36,+>,[3/2>) 7/2> [3/2>
[7> = [([37,+>,[3/2>) 7/2> [3/2>
[8> = [([38,+>,[3/2>) 7/2> [3/2>
[9> = [([47,+>,[3/2>) 7/2> [3/2>
[10> = [([53,+>,[3/2>) 7/2> [3/2>
[11> = [([54,+>,[3/2>) 7/2> [3/2>
[12> = [([55,+>,[3/2>) 7/2> [3/2>
[13> = [([19,+>,[1/2>) 7/2> [3/2>
[14> = [([22,+>,[1/2>) 7/2> [3/2>
[15> = [([23,+>,[1/2>) 7/2> [3/2>
[16> = [([48,+>,[1/2>) 7/2> [3/2>
[17> = [([49,+>,[1/2>) 7/2> [3/2>
[18> = [([50,+>,[1/2>) 7/2> [3/2>
[19> = [([56,+>,[1/2>) 7/2> [3/2>
[20> = [([57,+>,[1/2>) 7/2> [3/2>
[21> = [([58,+>,[1/2>) 7/2> [3/2>
[22> = [([59,+>,[1/2>) 7/2> [3/2>
[23> = [([60,+>,[1/2>) 7/2> [3/2>

Δ 7/2-

[1> = [([7,->,[3/2>) 7/2> [3/2>
[2> = [([18,->,[3/2>) 7/2> [3/2>
[3> = [([22,->,[3/2>) 7/2> [3/2>
[4> = [([23,->,[3/2>) 7/2> [3/2>
[5> = [([30,->,[3/2>) 7/2> [3/2>
[6> = [([33,->,[3/2>) 7/2> [3/2>
[7> = [([8,->,[1/2>) 7/2> [3/2>
[8> = [([24,->,[1/2>) 7/2> [3/2>
[9> = [([25,->,[1/2>) 7/2> [3/2>
[10> = [([26,->,[1/2>) 7/2> [3/2>
[11> = [([27,->,[1/2>) 7/2> [3/2>
[12> = [([31,->,[1/2>) 7/2> [3/2>

Δ 9/2+

[1> = [([21,+>,[3/2>) 9/2> [3/2>
[2> = [([47,+>,[3/2>) 9/2> [3/2>
[3> = [([53,+>,[3/2>) 9/2> [3/2>
[4> = [([54,+>,[3/2>) 9/2> [3/2>
[5> = [([55,+>,[3/2>) 9/2> [3/2>
[6> = [([66,+>,[3/2>) 9/2> [3/2>
[7> = [([67,+>,[3/2>) 9/2> [3/2>
[8> = [([22,+>,[1/2>) 9/2> [3/2>
[9> = [([23,+>,[1/2>) 9/2> [3/2>
[10> = [([56,+>,[1/2>) 9/2> [3/2>
[11> = [([57,+>,[1/2>) 9/2> [3/2>
[12> = [([58,+>,[1/2>) 9/2> [3/2>
[13> = [([59,+>,[1/2>) 9/2> [3/2>
[14> = [([60,+>,[1/2>) 9/2> [3/2>
[15> = [([63,+>,[1/2>) 9/2> [3/2>
[16> = [([64,+>,[1/2>) 9/2> [3/2>

Table 10.2. Continued

Δ 9/2-

[1> = [([7,->,[3/2>) 9/2> [3/2>
[2> = [([22,->,[3/2>) 9/2> [3/2>
[3> = [([23,->,[3/2>) 9/2> [3/2>
[4> = [([30,->,[3/2>) 9/2> [3/2>
[5> = [([33,->,[3/2>) 9/2> [3/2>
[6> = [([31,->,[1/2>) 9/2> [3/2>
[7> = [([34,->,[1/2>) 9/2> [3/2>
[8> = [([35,->,[1/2>) 9/2> [3/2>

Δ 11/2+

[1> = [([21,+>,[3/2>) 11/2> [3/2>
[2> = [([53,+>,[3/2>) 11/2> [3/2>
[3> = [([54,+>,[3/2>) 11/2> [3/2>
[4> = [([55,+>,[3/2>) 11/2> [3/2>
[5> = [([66,+>,[3/2>) 11/2> [3/2>
[6> = [([67,+>,[3/2>) 11/2> [3/2>
[7> = [([63,+>,[1/2>) 11/2> [3/2>
[8> = [([64,+>,[1/2>) 11/2> [3/2>
[9> = [([68,+>,[1/2>) 11/2> [3/2>
[10> = [([69,+>,[1/2>) 11/2> [3/2>

Δ 11/2-

[1> = [([30,->,[3/2>) 11/2> [3/2>
[2> = [([33,->,[3/2>) 11/2> [3/2>
[3> = [([34,->,[1/2>) 11/2> [3/2>
[4> = [([35,->,[1/2>) 11/2> [3/2>

Δ 13/2+

[1> = [([66,+>,[3/2>) 13/2> [3/2>
[2> = [([67,+>,[3/2>) 13/2> [3/2>
[3> = [([68,+>,[1/2>) 13/2> [3/2>
[4> = [([69,+>,[1/2>) 13/2> [3/2>

Δ 13/2-

[1> = [([33,->,[3/2>) 13/2> [3/2>

Δ 15/2+

[1> = [([66,+>,[3/2>) 15/2> [3/2>
[2> = [([67,+>,[3/2>) 15/2> [3/2>

C. Table of the Eigenvectors

Table 10.3 is a listing of the N and Δ eigenvectors resulting from the diagonalization of H in a 6ω basis. The notation is

$$[M_R> = \sum_I e_I [I> \quad ,$$

where M_R is the mass eigenvalue of the resonance in the 6ω basis, as given in Table 4.3. The states [I> are the basis functions of Table 10.2 and e_I is the coefficient of [I> in the eigenvector.

Table 10.3. The eigenvectors corresponding to the 6ω eigenvalues of Table 4.3

N 1/2+

[954>= (-0.5164)[1>+(-0.5849)[2>+(-0.4751)[3>+(-0.1877)[4>+(-0.2901)[5>
+(-0.0893)[6>+(-0.1637)[7>+(-0.0493)[8>+(-0.0135)[9>+(0.0630)[10>
+(-0.0226)[11>+(0.0029)[12>+(0.0026)[13>+(0.0156)[14>+(-0.0065)[15>
+(0.0090)[16>+(0.0552)[17>+(-0.0222)[18>+(0.0132)[19>+(0.0027)[20>
+(0.0022)[21>+(0.0030)[22>+(0.0026)[23>+(0.0136)[24>+(-0.0065)[25>
+(0.0083)[26>+(-0.0038)[27>+(-0.0035)[28>+(-0.0002)[29>+(0.0026)[30>
+(0.0030)[31>+(-0.0001)[32>+(0.0021)[33>+(0.0009)[34>

[1583>= (-0.0048)[1>+(0.0169)[2>+(0.0140)[3>+(0.0253)[4>+(0.0025)[5>
+(0.0055)[6>+(0.0248)[7>+(-0.2642)[8>+(0.4681)[9>+(0.2609)[10>
+(-0.1249)[11>+(0.0328)[12>+(-0.0173)[13>+(-0.3672)[14>+(0.2396)[15>
+(-0.2853)[16>+(0.1729)[17>+(-0.1041)[18>+(0.0404)[19>+(0.0233)[20>
+(-0.0112)[21>+(0.0282)[22>+(-0.0161)[23>+(-0.2134)[24>+(0.1947)[25>
+(-0.1827)[26>+(0.1203)[27>+(0.1089)[28>+(0.0134)[29>+(0.2396)[30>
+(0.2649)[31>+(-0.0047)[32>+(0.1735)[33>+(0.0771)[34>

[1689>= (0.0447)[1>+(-0.0680)[2>+(-0.0845)[3>+(-0.0775)[4>+(-0.0447)[5>
+(-0.0293)[6>+(-0.0751)[7>+(0.5117)[8>+(0.3094)[9>+(-0.5082)[10>
+(0.1923)[11>+(0.0011)[12>+(-0.0032)[13>+(-0.2349)[14>+(0.1496)[15>
+(-0.1917)[16>+(-0.3365)[17>+(0.1490)[18>+(-0.0682)[19>+(0.0012)[20>
+(0.0017)[21>+(0.0005)[22>+(-0.0035)[23>+(-0.1302)[24>+(0.1175)[25>
+(-0.1218)[26>+(0.0754)[27>+(0.0722)[28>+(0.0078)[29>+(-0.0551)[30>
+(-0.0596)[31>+(-0.0008)[32>+(-0.0380)[33>+(-0.0167)[34>

[1822>= (0.0504)[1>+(-0.0156)[2>+(-0.0277)[3>+(-0.0397)[4>+(-0.0126)[5>
+(-0.0126)[6>+(-0.0375)[7>+(0.1770)[8>+(-0.1915)[9>+(-0.1809)[10>
+(0.0173)[11>+(0.0053)[12>+(0.0033)[13>+(0.1584)[14>+(-0.0506)[15>
+(0.1203)[16>+(-0.1225)[17>+(0.0024)[18>+(-0.0160)[19>+(0.0017)[20>
+(0.0036)[21>+(0.0054)[22>+(0.0037)[23>+(0.0950)[24>+(-0.0326)[25>
+(0.0809)[26>+(-0.0321)[27>+(-0.0440)[28>+(-0.0022)[29>+(0.5483)[30>
+(0.5911)[31>+(-0.0050)[32>+(0.3687)[33>+(0.1714)[34>

N 1/2-

[1357>= (0.0133)[1>+(0.0128)[2>+(-0.0069)[3>+(0.3076)[4>+(0.5270)[5>
+(0.3003)[6>+(0.5197)[7>+(0.1230)[8>+(0.2047)[9>+(-0.0069)[10>
+(0.1906)[11>+(0.3297)[12>+(0.0449)[13>+(0.0671)[14>+(-0.0869)[15>
+(-0.1538)[16>+(0.0792)[17>+(0.1363)[18>+(-0.0085)[19>+(-0.0020)[20>
+(-0.0331)[21>+(-0.0357)[22>+(0.0086)[23>

[1486>= (-0.0584)[1>+(-0.0565)[2>+(0.0193)[3>+(-0.5414)[4>+(0.3153)[5>
+(-0.4914)[6>+(0.2905)[7>+(-0.2544)[8>+(0.1437)[9>+(-0.0053)[10>
+(-0.2911)[11>+(0.1718)[12>+(-0.1178)[13>+(0.0613)[14>+(0.1488)[15>
+(-0.0906)[16>+(-0.1415)[17>+(0.0835)[18>+(-0.0031)[19>+(-0.0039)[20>
+(0.0100)[21>+(0.0106)[22>+(-0.0036)[23>

[2080>= (-0.7256)[1>+(-0.5206)[2>+(0.1963)[3>+(-0.0161)[4>+(-0.0103)[5>
+(0.1896)[6>+(-0.0389)[7>+(-0.2132)[8>+(0.0059)[9>+(-0.0997)[10>
+(0.1424)[11>+(-0.0313)[12>+(-0.1806)[13>+(0.0051)[14>+(-0.0220)[15>
+(0.0027)[16>+(0.0097)[17>+(-0.0102)[18>+(-0.0686)[19>+(-0.0366)[20>
+(-0.0790)[21>+(-0.0624)[22>+(0.0112)[23>

[2149>= (0.1111)[1>+(0.0802)[2>+(-0.0234)[3>+(-0.0068)[4>+(-0.0187)[5>
+(0.0312)[6>+(0.1172)[7>+(-0.1823)[8>+(-0.3428)[9>+(0.1166)[10>
+(0.0263)[11>+(0.0988)[12>+(-0.1304)[13>+(-0.2477)[14>+(0.0098)[15>
+(0.0179)[16>+(-0.0692)[17>+(-0.0972)[18>+(0.0780)[19>+(0.0487)[20>
+(-0.6556)[21>+(-0.4977)[22>+(0.1102)[23>

Table 10.3. Continued

N 3/2+

[1676]= (0.2314)[1]+(0.2356)[2]+(0.0809)[3]+(0.1465)[4]+(0.0538)[5]
+(0.0584)[6]+(0.0514)[7]+(-0.2270)[8]+(-0.3889)[9]+(-0.3024)[10]
+(0.2129)[11]+(-0.1197)[12]+(0.0113)[13]+(0.0405)[14]+(0.3015)[15]
+(0.2211)[16]+(-0.1466)[17]+(-0.1508)[18]+(0.2571)[19]+(0.1959)[20]
+(0.0092)[21]+(0.1334)[22]+(-0.0996)[23]+(0.0344)[24]+(0.0077)[25]
+(0.0256)[26]+(0.0093)[27]+(0.0362)[28]+(0.1714)[29]+(0.1176)[30]
+(-0.1074)[31]+(-0.1175)[32]+(0.1703)[33]+(0.1229)[34]+(-0.0836)[35]
+(-0.0812)[36]+(-0.1041)[37]+(-0.0763)[38]+(-0.0100)[39]+(-0.0089)[40]
+(0.0067)[41]+(0.0071)[42]+(-0.0016)[43]+(0.1118)[44]+(0.1214)[45]
+(-0.0278)[46]+(0.0780)[47]+(0.0344)[48]+(-0.0259)[49]+(0.0147)[50]

[1725]= (0.4040)[1]+(0.4134)[2]+(0.1255)[3]+(0.2574)[4]+(0.0922)[5]
+(0.0961)[6]+(0.0720)[7]+(0.0677)[8]+(-0.1320)[9]+(0.4253)[10]
+(-0.0603)[11]+(0.0389)[12]+(0.0014)[13]+(-0.0272)[14]+(0.1152)[15]
+(-0.3367)[16]+(-0.0114)[17]+(0.1847)[18]+(0.0910)[19]+(-0.2542)[20]
+(-0.0041)[21]+(-0.0360)[22]+(0.0323)[23]+(-0.0098)[24]+(0.0008)[25]
+(-0.0170)[26]+(0.0014)[27]+(-0.0236)[28]+(0.0728)[29]+(-0.1960)[30]
+(-0.0020)[31]+(0.1414)[32]+(0.0672)[33]+(-0.1644)[34]+(-0.0103)[35]
+(0.0921)[36]+(-0.0387)[37]+(0.0927)[38]+(-0.0012)[39]+(0.0117)[40]
+(-0.0057)[41]+(-0.0016)[42]+(-0.0019)[43]+(0.0200)[44]+(0.0226)[45]
+(0.0115)[46]+(0.0145)[47]+(0.0071)[48]+(0.0111)[49]+(-0.0055)[50]

[1757]= (0.1310)[1]+(0.1284)[2]+(0.0502)[3]+(0.0755)[4]+(0.0305)[5]
+(0.0349)[6]+(0.0321)[7]+(0.5249)[8]+(-0.0951)[9]+(-0.2232)[10]
+(-0.5027)[11]+(0.2369)[12]+(-0.0038)[13]+(-0.0268)[14]+(0.0691)[15]
+(0.1813)[16]+(-0.0247)[17]+(-0.0948)[18]+(0.0716)[19]+(0.1309)[20]
+(0.0033)[21]+(-0.3189)[22]+(0.1885)[23]+(-0.0769)[24]+(-0.0036)[25]
+(-0.0184)[26]+(-0.0031)[27]+(-0.0225)[28]+(0.0354)[29]+(0.1079)[30]
+(-0.0156)[31]+(-0.0729)[32]+(0.0487)[33]+(0.0839)[34]+(-0.0179)[35]
+(-0.0477)[36]+(-0.0299)[37]+(-0.0474)[38]+(-0.0017)[39]+(-0.0036)[40]
+(0.0026)[41]+(0.0022)[42]+(-0.0007)[43]+(-0.1546)[44]+(-0.1656)[45]
+(0.0002)[46]+(-0.1042)[47]+(-0.0463)[48]+(0.0006)[49]+(-0.0001)[50]

[1823]= (-0.3652)[1]+(-0.3610)[2]+(-0.1148)[3]+(-0.2165)[4]+(-0.0744)[5]
+(-0.0843)[6]+(-0.0648)[7]+(0.1585)[8]+(-0.3907)[9]+(0.1848)[10]
+(-0.1515)[11]+(0.0621)[12]+(0.0050)[13]+(0.0055)[14]+(0.2938)[15]
+(-0.1389)[16]+(-0.1797)[17]+(0.0829)[18]+(0.2394)[19]+(-0.1160)[20]
+(0.0021)[21]+(-0.0957)[22]+(0.0472)[23]+(-0.0205)[24]+(0.0026)[25]
+(0.0056)[26]+(0.0049)[27]+(0.0021)[28]+(0.1605)[29]+(-0.0758)[30]
+(-0.1352)[31]+(0.0626)[32]+(0.1489)[33]+(-0.0724)[34]+(-0.0919)[35]
+(0.0428)[36]+(-0.0875)[37]+(0.0440)[38]+(-0.0072)[39]+(0.0019)[40]
+(0.0026)[41]+(0.0012)[42]+(0.0007)[43]+(0.1856)[44]+(0.1987)[45]
+(-0.0116)[46]+(0.1236)[47]+(0.0567)[48]+(-0.0108)[49]+(0.0056)[50]

N 3/2-

[1531]= (-0.0339)[1]+(-0.0325)[2]+(0.0132)[3]+(-0.0020)[4]+(-0.4418)[5]
+(-0.4522)[6]+(-0.3991)[7]+(-0.4251)[8]+(-0.2009)[9]+(-0.1785)[10]
+(-0.0052)[11]+(-0.0030)[12]+(0.0060)[13]+(-0.2347)[14]+(-0.2567)[15]
+(-0.0909)[16]+(-0.0635)[17]+(0.1179)[18]+(0.1193)[19]+(-0.1111)[20]
+(-0.1101)[21]+(-0.0055)[22]+(-0.0016)[23]+(-0.0017)[24]+(-0.0025)[25]
+(0.0061)[26]+(0.0022)[27]+(-0.0017)[28]+(0.0013)[29]+(0.0041)[30]
+(0.0041)[31]+(-0.0014)[32]+(0.0022)[33]

[1577]= (0.0429)[1]+(0.0394)[2]+(-0.0166)[3]+(-0.0007)[4]+(0.4584)[5]
+(-0.4454)[6]+(0.3989)[7]+(-0.4054)[8]+(0.2220)[9]+(-0.1889)[10]
+(-0.0048)[11]+(0.0165)[12]+(0.0056)[13]+(0.2269)[14]+(-0.2373)[15]
+(0.1076)[16]+(-0.0745)[17]+(-0.1183)[18]+(0.1152)[19]+(0.1163)[20]
+(-0.1103)[21]+(-0.0055)[22]+(0.0152)[23]+(-0.0015)[24]+(0.0080)[25]
+(0.0053)[26]+(-0.0011)[27]+(-0.0019)[28]+(0.0013)[29]+(0.0221)[30]
+(0.0231)[31]+(-0.0071)[32]+(-0.0012)[33]

[2113]= (0.7607)[1]+(0.5331)[2]+(-0.2174)[3]+(0.0038)[4]+(0.0124)[5]
+(0.0073)[6]+(-0.1622)[7]+(0.0310)[8]+(0.1325)[9]+(-0.0530)[10]
+(0.0162)[11]+(0.0459)[12]+(0.0300)[13]+(-0.1135)[14]+(0.0276)[15]
+(0.1266)[16]+(-0.0339)[17]+(0.0362)[18]+(0.0141)[19]+(-0.0309)[20]
+(-0.0099)[21]+(0.0092)[22]+(0.0256)[23]+(0.0054)[24]+(0.0241)[25]
+(0.0242)[26]+(0.0057)[27]+(-0.0018)[28]+(0.0061)[29]+(-0.0472)[30]
+(-0.0362)[31]+(0.0111)[32]+(0.0026)[33]

Table 10.3. Continued

```
[2183>= (-0.1336)[ 1>+(-0.0924)[ 2>+( 0.0367)[ 3>+(-0.0094)[ 4>+( 0.0154)[ 5>
       +(-0.0213)[ 6>+(-0.1027)[ 7>+( 0.1041)[ 8>+( 0.1977)[ 9>+(-0.1981)[10>
       +(-0.2059)[11>+( 0.2971)[12>+( 0.4914)[13>+(-0.0987)[14>+( 0.0958)[15>
       +( 0.1326)[16>+(-0.1380)[17>+(-0.0331)[18>+( 0.0230)[19>+( 0.0406)[20>
       +(-0.0372)[21>+(-0.1325)[22>+( 0.1956)[23>+(-0.0890)[24>+( 0.1250)[25>
       +( 0.3453)[26>+(-0.0052)[27>+(-0.1058)[28>+( 0.1092)[29>+(-0.3690)[30>
       +(-0.2694)[31>+( 0.0956)[32>+( 0.0201)[33>

                                N 5/2+

[1794>= (-0.5261)[ 1>+(-0.5183)[ 2>+(-0.1786)[ 3>+(-0.3094)[ 4>+(-0.1174)[ 5>
       +(-0.1246)[ 6>+(-0.1088)[ 7>+(-0.0002)[ 8>+( 0.3147)[ 9>+( 0.0817)[10>
       +(-0.0072)[11>+(-0.2305)[12>+(-0.0534)[13>+( 0.0760)[14>+( 0.0407)[15>
       +(-0.2334)[16>+(-0.0609)[17>+(-0.0084)[18>+(-0.0094)[19>+( 0.0005)[20>
       +(-0.0018)[21>+(-0.0059)[22>+(-0.0064)[23>+(-0.1218)[24>+(-0.0237)[25>
       +( 0.0446)[26>+( 0.0303)[27>+(-0.1550)[28>+(-0.0368)[29>+( 0.0575)[30>
       +( 0.0267)[31>+( 0.1005)[32>+( 0.0246)[33>+( 0.0034)[34>+( 0.0006)[35>
       +(-0.0041)[36>+(-0.0069)[37>+(-0.0049)[38>+(-0.0069)[39>+( 0.0022)[40>
       +( 0.0019)[41>+( 0.0010)[42>+(-0.0015)[43>+(-0.0010)[44>+(-0.0010)[45>
       +( 0.0005)[46>+(-0.0042)[47>+( 0.0054)[48>+(-0.0033)[49>+( 0.0028)[50>
       +( 0.0043)[51>+( 0.0029)[52>

[1845>= ( 0.2200)[ 1>+( 0.2209)[ 2>+( 0.0691)[ 3>+( 0.1343)[ 4>+( 0.0445)[ 5>
       +( 0.0521)[ 6>+( 0.0390)[ 7>+( 0.0013)[ 8>+( 0.2231)[ 9>+( 0.5325)[10>
       +(-0.0275)[11>+(-0.1643)[12>+(-0.4059)[13>+( 0.0961)[14>+( 0.2089)[15>
       +(-0.1388)[16>+(-0.3269)[17>+( 0.0083)[18>+(-0.0003)[19>+( 0.0086)[20>
       +( 0.0092)[21>+(-0.0186)[22>+(-0.0286)[23>+(-0.0875)[24>+(-0.2250)[25>
       +( 0.0710)[26>+( 0.1510)[27>+(-0.0850)[28>+(-0.2066)[29>+( 0.0511)[30>
       +( 0.1095)[31>+( 0.0519)[32>+( 0.1198)[33>+( 0.0042)[34>+( 0.0112)[35>
       +( 0.0066)[36>+( 0.0004)[37>+( 0.0052)[38>+(-0.0009)[39>+(-0.0016)[40>
       +(-0.0002)[41>+( 0.0069)[42>+( 0.0061)[43>+( 0.0080)[44>+( 0.0051)[45>
       +(-0.0012)[46>+( 0.0090)[47>+( 0.0027)[48>+( 0.0077)[49>+(-0.0049)[50>
       +( 0.0019)[51>+( 0.0018)[52>

[1907>= (-0.2482)[ 1>+(-0.2465)[ 2>+(-0.0692)[ 3>+(-0.1472)[ 4>+(-0.0678)[ 5>
       +(-0.0548)[ 6>+(-0.0346)[ 7>+( 0.0014)[ 8>+(-0.4888)[ 9>+( 0.3047)[10>
       +(-0.0086)[11>+( 0.3736)[12>+(-0.2348)[13>+(-0.1591)[14>+( 0.1069)[15>
       +( 0.0017)[16>+(-0.1843)[17>+( 0.0009)[18>+(-0.0113)[19>+( 0.0005)[20>
       +(-0.0148)[21>+(-0.0064)[22>+(-0.0063)[23>+( 0.2074)[24>+(-0.1314)[25>
       +(-0.1076)[26>+( 0.0738)[27>+( 0.1920)[28>+(-0.1178)[29>+(-0.0856)[30>
       +( 0.0560)[31>+(-0.1108)[32>+( 0.0650)[33>+(-0.0085)[34>+( 0.0062)[35>
       +( 0.0014)[36>+(-0.0110)[37>+(-0.0001)[38>+(-0.0052)[39>+( 0.0004)[40>
       +( 0.0002)[41>+( 0.0006)[42>+(-0.0124)[43>+(-0.0044)[44>+(-0.0083)[45>
       +( 0.0008)[46>+( 0.0205)[47>+(-0.0074)[48>+( 0.0172)[49>+(-0.0115)[50>
       +(-0.0064)[51>+(-0.0036)[52>

[2353>= (-0.0249)[ 1>+( 0.0930)[ 2>+(-0.0650)[ 3>+( 0.0891)[ 4>+(-0.0242)[ 5>
       +( 0.0313)[ 6>+(-0.0577)[ 7>+(-0.0001)[ 8>+( 0.0722)[ 9>+( 0.0130)[10>
       +( 0.0710)[11>+( 0.0927)[12>+(-0.0708)[13>+(-0.6250)[14>+(-0.1601)[15>
       +(-0.3563)[16>+(-0.0396)[17>+(-0.0456)[18>+( 0.0430)[19>+(-0.1688)[20>
       +(-0.0196)[21>+( 0.0334)[22>+( 0.0454)[23>+( 0.0988)[24>+(-0.0578)[25>
       +(-0.4615)[26>+(-0.1110)[27>+(-0.1975)[28>+(-0.0442)[29>+(-0.1296)[30>
       +(-0.0426)[31>+( 0.1881)[32>+( 0.0174)[33>+(-0.0224)[34>+(-0.0107)[35>
       +(-0.0270)[36>+( 0.0253)[37>+(-0.0244)[38>+( 0.0257)[39>+( 0.0093)[40>
       +(-0.0046)[41>+(-0.1113)[42>+( 0.0160)[43>+(-0.0739)[44>+(-0.0152)[45>
       +(-0.0080)[46>+(-0.0359)[47>+( 0.0861)[48>+(-0.0134)[49>+( 0.0256)[50>
       +( 0.0576)[51>+( 0.0335)[52>

                                N 5/2-

[1685>= ( 0.0156)[ 1>+( 0.0015)[ 2>+( 0.0128)[ 3>+(-0.0097)[ 4>+(-0.6570)[ 5>
       +(-0.5639)[ 6>+(-0.2941)[ 7>+( 0.0100)[ 8>+(-0.0060)[ 9>+(-0.0019)[10>
       +( 0.0188)[11>+(-0.3104)[12>+(-0.1278)[13>+( 0.1612)[14>+(-0.1550)[15>
       +( 0.0080)[16>+(-0.0065)[17>+( 0.0058)[18>+(-0.0017)[19>+(-0.0018)[20>
       +( 0.0149)[21>+(-0.0028)[22>+( 0.0019)[23>+( 0.0003)[24>+(-0.0053)[25>
       +(-0.0005)[26>+( 0.0033)[27>+(-0.0002)[28>+(-0.0029)[29>+( 0.0021)[30>
       +(-0.0030)[31>+( 0.0015)[32>
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Table 10.3. Continued

```
[2202>= (-0.2393)[ 1>+(-0.0044)[ 2>+(-0.1085)[ 3>+( 0.1489)[ 4>+(-0.0175)[ 5>
       +( 0.0918)[ 6>+(-0.1647)[ 7>+( 0.1080)[ 8>+( 0.2209)[ 9>+( 0.5750)[10>
       +( 0.2242)[11>+( 0.0874)[12>+(-0.1090)[13>+( 0.0307)[14>+(-0.0284)[15>
       +( 0.0661)[16>+( 0.1391)[17>+( 0.0535)[18>+( 0.1015)[19>+( 0.3811)[20>
       +( 0.1582)[21>+(-0.0096)[22>+(-0.0115)[23>+(-0.1622)[24>+(-0.0426)[25>
       +( 0.1312)[26>+( 0.0479)[27>+( 0.0033)[28>+( 0.3106)[29>+(-0.0163)[30>
       +( 0.1664)[31>+(-0.1767)[32>

[2228>= ( 0.3502)[ 1>+( 0.0044)[ 2>+( 0.1683)[ 3>+(-0.2048)[ 4>+(-0.0255)[ 5>
       +( 0.1492)[ 6>+(-0.2074)[ 7>+( 0.0921)[ 8>+( 0.1840)[ 9>+(-0.3495)[10>
       +( 0.4676)[11>+( 0.1313)[12>+(-0.1433)[13>+( 0.0249)[14>+(-0.0182)[15>
       +( 0.0581)[16>+( 0.1140)[17>+( 0.0403)[18>+( 0.0874)[19>+(-0.2172)[20>
       +( 0.3175)[21>+(-0.0040)[22>+( 0.0056)[23>+( 0.1266)[24>+(-0.1154)[25>
       +(-0.0809)[26>+( 0.0993)[27>+(-0.0071)[28>+( 0.2509)[29>+(-0.0065)[30>
       +( 0.1374)[31>+(-0.1355)[32>

[2267>= (-0.0112)[ 1>+( 0.0017)[ 2>+(-0.0028)[ 3>+( 0.0092)[ 4>+( 0.0526)[ 5>
       +(-0.1882)[ 6>+( 0.3305)[ 7>+(-0.1813)[ 8>+(-0.1604)[ 9>+( 0.1457)[10>
       +( 0.5803)[11>+(-0.1691)[12>+( 0.2248)[13>+(-0.0410)[14>+( 0.0449)[15>
       +(-0.1155)[16>+(-0.1000)[17>+(-0.0742)[18>+(-0.0719)[19>+( 0.0909)[20>
       +( 0.3797)[21>+(-0.0096)[22>+(-0.0146)[23>+(-0.0560)[24>+(-0.1717)[25>
       +( 0.0256)[26>+( 0.1161)[27>+(-0.0016)[28>+(-0.2759)[29>+( 0.0132)[30>
       +(-0.1562)[31>+( 0.1381)[32>

                                N 7/2+

[1982>= ( 0.0063)[ 1>+(-0.0001)[ 2>+( 0.0048)[ 3>+( 0.0042)[ 4>+( 0.0018)[ 5>
       +( 0.6385)[ 6>+(-0.4673)[ 7>+( 0.2036)[ 8>+(-0.3986)[ 9>+( 0.0005)[10>
       +( 0.0071)[11>+(-0.0022)[12>+( 0.0111)[13>+(-0.0137)[14>+( 0.0016)[15>
       +(-0.2471)[16>+( 0.1321)[17>+(-0.2452)[18>+( 0.1134)[19>+( 0.1460)[20>
       +( 0.0080)[21>+(-0.0016)[22>+( 0.0068)[23>+( 0.0032)[24>+( 0.0036)[25>
       +(-0.0010)[26>+( 0.0008)[27>+(-0.0020)[28>+( 0.0088)[29>+(-0.0106)[30>
       +(-0.0006)[31>+(-0.0052)[32>+( 0.0071)[33>+(-0.0078)[34>+( 0.0003)[35>
       +( 0.0008)[36>+(-0.0014)[37>+( 0.0002)[38>+( 0.0001)[39>+( 0.0091)[40>
       +( 0.0075)[41>+( 0.0045)[42>+(-0.0011)[43>+( 0.0003)[44>

[2403>= ( 0.5289)[ 1>+(-0.0010)[ 2>+( 0.3753)[ 3>+( 0.0844)[ 4>+( 0.1313)[ 5>
       +( 0.0092)[ 6>+( 0.0953)[ 7>+(-0.0535)[ 8>+(-0.1060)[ 9>+(-0.0335)[10>
       +(-0.0791)[11>+( 0.3886)[12>+( 0.2831)[13>+(-0.2839)[14>+( 0.0267)[15>
       +( 0.0808)[16>+(-0.0574)[17>+(-0.0331)[18>+( 0.0239)[19>+( 0.0524)[20>
       +(-0.0116)[21>+(-0.0193)[22>+(-0.0373)[23>+(-0.0167)[24>+(-0.0528)[25>
       +( 0.0059)[26>+( 0.0195)[27>+( 0.2569)[28>+( 0.1781)[29>+(-0.1814)[30>
       +(-0.0274)[31>+( 0.0976)[32>+( 0.1335)[33>+(-0.1703)[34>+( 0.0145)[35>
       +( 0.0087)[36>+( 0.0020)[37>+( 0.0076)[38>+( 0.0024)[39>+( 0.0520)[40>
       +( 0.0348)[41>+( 0.0199)[42>+(-0.0008)[43>+( 0.0176)[44>

[2428>= (-0.4452)[ 1>+( 0.0040)[ 2>+(-0.3155)[ 3>+(-0.0736)[ 4>+(-0.1037)[ 5>
       +( 0.0389)[ 6>+( 0.1608)[ 7>+(-0.1680)[ 8>+(-0.2465)[ 9>+(-0.0150)[10>
       +(-0.1881)[11>+( 0.2276)[12>+( 0.4240)[13>+( 0.1590)[14>+(-0.0952)[15>
       +( 0.1422)[16>+(-0.1535)[17>+(-0.0957)[18>+( 0.0312)[19>+( 0.1179)[20>
       +(-0.0230)[21>+(-0.0040)[22>+(-0.1010)[23>+(-0.0139)[24>+(-0.1078)[25>
       +( 0.0064)[26>+( 0.0346)[27>+( 0.1411)[28>+( 0.2745)[29>+( 0.0378)[30>
       +(-0.1098)[31>+( 0.1345)[32>+( 0.1497)[33>+( 0.0978)[34>+(-0.0459)[35>
       +( 0.0054)[36>+( 0.0097)[37>+(-0.0016)[38>+( 0.0025)[39>+( 0.0978)[40>
       +( 0.0654)[41>+( 0.0384)[42>+( 0.0061)[43>+( 0.0246)[44>

[2442>= (-0.1159)[ 1>+( 0.0042)[ 2>+(-0.0796)[ 3>+(-0.0314)[ 4>+(-0.0239)[ 5>
       +(-0.0826)[ 6>+(-0.0734)[ 7>+( 0.6309)[ 8>+( 0.3398)[ 9>+(-0.0172)[10>
       +(-0.0342)[11>+( 0.1865)[12>+( 0.2210)[13>+(-0.0096)[14>+(-0.1039)[15>
       +(-0.0823)[16>+( 0.4574)[17>+( 0.1882)[18>+( 0.1301)[19>+(-0.1747)[20>
       +( 0.0139)[21>+(-0.0052)[22>+(-0.0186)[23>+(-0.0120)[24>+(-0.0172)[25>
       +( 0.0056)[26>+( 0.0055)[27>+( 0.1167)[28>+( 0.1448)[29>+(-0.0303)[30>
       +(-0.0897)[31>+( 0.0798)[32>+( 0.0646)[33>+(-0.0074)[34>+(-0.0488)[35>
       +(-0.0018)[36>+( 0.0066)[37>+( 0.0004)[38>+( 0.0021)[39>+(-0.0168)[40>
       +(-0.0098)[41>+(-0.0063)[42>+(-0.0024)[43>+( 0.0075)[44>
```

Table 10.3. Continued

N 7/2-

[2275]= (-0.2285)[1]+(-0.0872)[2]+(0.1523)[3]+(-0.0017)[4]+(0.0247)[5]
+(0.7634)[6]+(-0.0394)[7]+(0.0160)[8]+(0.0078)[9]+(0.4779)[10]
+(-0.0247)[11]+(-0.0183)[12]+(-0.0108)[13]+(-0.2417)[14]+(0.0152)[15]
+(0.1736)[16]+(-0.0070)[17]+(0.0020)[18]+(0.0079)[19]+(-0.0002)[20]
+(0.0039)[21]+(-0.0918)[22]+(-0.0464)[23]+(0.0551)[24]+(0.0050)[25]

[2296]= (-0.1157)[1]+(-0.0589)[2]+(0.0611)[3]+(0.0064)[4]+(-0.1036)[5]
+(-0.0886)[6]+(-0.7720)[7]+(-0.0611)[8]+(-0.0515)[9]+(-0.0595)[10]
+(-0.5001)[11]+(0.0085)[12]+(0.0140)[13]+(0.0206)[14]+(0.2168)[15]
+(-0.0202)[16]+(-0.1582)[17]+(-0.0003)[18]+(0.0013)[19]+(0.0046)[20]
+(-0.0080)[21]+(-0.1414)[22]+(-0.0729)[23]+(0.0808)[24]+(-0.0018)[25]

[2346]= (0.7416)[1]+(0.3545)[2]+(-0.4093)[3]+(0.0003)[4]+(-0.0843)[5]
+(0.1949)[6]+(-0.0814)[7]+(-0.0528)[8]+(-0.0339)[9]+(0.1336)[10]
+(-0.0512)[11]+(0.0080)[12]+(0.0098)[13]+(-0.0428)[14]+(0.0196)[15]
+(0.0344)[16]+(-0.0155)[17]+(-0.0052)[18]+(-0.0004)[19]+(-0.0007)[20]
+(0.0079)[21]+(-0.2225)[22]+(-0.1182)[23]+(0.1165)[24]+(0.0010)[25]

[2367]= (0.1872)[1]+(0.0892)[2]+(-0.1020)[3]+(0.0024)[4]+(0.2540)[5]
+(0.1230)[6]+(-0.2082)[7]+(0.1591)[8]+(0.1008)[9]+(0.0768)[10]
+(-0.1243)[11]+(0.0025)[12]+(0.0055)[13]+(-0.0438)[14]+(0.0802)[15]
+(0.0221)[16]+(-0.0363)[17]+(-0.0041)[18]+(0.0032)[19]+(0.0048)[20]
+(0.0051)[21]+(0.7003)[22]+(0.3780)[23]+(-0.3515)[24]+(0.0015)[25]

N 9/2+

[2452]= (0.6573)[1]+(0.4600)[2]+(0.0937)[3]+(0.1628)[4]+(-0.0237)[5]
+(0.3479)[6]+(0.1370)[7]+(-0.2481)[8]+(0.0096)[9]+(-0.0110)[10]
+(-0.0173)[11]+(0.0079)[12]+(0.2230)[13]+(0.0842)[14]+(-0.1437)[15]
+(-0.0138)[16]+(0.0988)[17]+(0.0676)[18]+(-0.1635)[19]+(0.0033)[20]
+(-0.0010)[21]+(-0.0030)[22]+(0.0053)[23]+(0.0061)[24]+(0.0024)[25]
+(0.0034)[26]+(0.0005)[27]+(0.0010)[28]+(0.0030)[29]+(-0.0016)[30]
+(-0.0044)[31]

[2500]= (-0.3609)[1]+(-0.2512)[2]+(-0.0636)[3]+(-0.0818)[4]+(-0.0939)[5]
+(0.3461)[6]+(0.5069)[7]+(-0.1886)[8]+(-0.2946)[9]+(-0.0513)[10]
+(-0.0509)[11]+(0.0178)[12]+(0.2200)[13]+(0.3250)[14]+(-0.1388)[15]
+(-0.2121)[16]+(0.1016)[17]+(0.1401)[18]+(-0.0998)[19]+(-0.1563)[20]
+(0.0052)[21]+(0.0059)[22]+(-0.0045)[23]+(0.0009)[24]+(-0.0057)[25]
+(-0.0045)[26]+(-0.0067)[27]+(-0.0048)[28]+(-0.0011)[29]+(0.0037)[30]
+(-0.0048)[31]

[2527]= (-0.2636)[1]+(-0.1817)[2]+(-0.0530)[3]+(-0.0532)[4]+(0.0451)[5]
+(0.3850)[6]+(-0.4503)[7]+(-0.4030)[8]+(0.2403)[9]+(0.0244)[10]
+(0.0241)[11]+(-0.0113)[12]+(0.2500)[13]+(-0.2891)[14]+(-0.2357)[15]
+(0.1705)[16]+(0.0615)[17]+(-0.1235)[18]+(-0.2241)[19]+(0.1321)[20]
+(0.0021)[21]+(0.0003)[22]+(-0.0009)[23]+(0.0009)[24]+(0.0030)[25]
+(-0.0080)[26]+(0.0008)[27]+(-0.0051)[28]+(0.0075)[29]+(-0.0124)[30]
+(-0.0048)[31]

[2574]= (0.0680)[1]+(0.0497)[2]+(-0.0011)[3]+(0.0154)[4]+(0.1672)[5]
+(-0.1484)[6]+(-0.2882)[7]+(-0.2367)[8]+(-0.6671)[9]+(0.0823)[10]
+(0.0973)[11]+(-0.0198)[12]+(-0.0918)[13]+(-0.1758)[14]+(-0.0956)[15]
+(-0.2990)[16]+(-0.1018)[17]+(-0.2499)[18]+(-0.1279)[19]+(-0.3391)[20]
+(-0.0078)[21]+(-0.0109)[22]+(-0.0005)[23]+(-0.0003)[24]+(-0.0032)[25]
+(0.0027)[26]+(-0.0001)[27]+(0.0010)[28]+(-0.0075)[29]+(-0.0071)[30]
+(-0.0031)[31]

N 9/2-

[2351]= (0.0047)[1]+(-0.0061)[2]+(0.8116)[3]+(0.5021)[4]+(-0.0160)[5]
+(-0.2414)[6]+(0.1746)[7]+(-0.0079)[8]+(0.0025)[9]+(0.0004)[10]
+(-0.0072)[11]+(-0.0069)[12]+(0.0072)[13]+(0.0050)[14]+(-0.0002)[15]

[2932]= (0.0249)[1]+(-0.4469)[2]+(-0.0021)[3]+(-0.0005)[4]+(-0.0415)[5]
+(0.0173)[6]+(0.0045)[7]+(0.0452)[8]+(0.0579)[9]+(0.8502)[10]
+(0.0126)[11]+(-0.2089)[12]+(0.0292)[13]+(0.0317)[14]+(0.1546)[15]

Table 10.3. Continued

|[2951]= (-0.0052)[1]+(-0.1998)[2]+(0.0158)[3]+(-0.1163)[4]+(0.1027)[5]
+(-0.1337)[6]+(0.0320)[7]+(-0.0550)[8]+(-0.1308)[9]+(-0.0320)[10]
+(-0.8613)[11]+(0.0010)[12]+(0.2816)[13]+(-0.0169)[14]+(-0.2729)[15]

|[2971]= (0.0077)[1]+(0.0317)[2]+(-0.0930)[3]+(0.5449)[4]+(-0.1247)[5]
+(0.7198)[6]+(-0.1617)[7]+(0.0992)[8]+(0.1307)[9]+(-0.0512)[10]
+(-0.2835)[11]+(-0.0403)[12]+(0.0085)[13]+(0.0319)[14]+(0.1454)[15]

N 11/2+

|[2563]= (-0.0060)[1]+(0.0052)[2]+(0.6362)[3]+(-0.4941)[4]+(0.4006)[5]
+(-0.3073)[6]+(0.1484)[7]+(-0.2723)[8]+(0.0044)[9]+(-0.0056)[10]
+(-0.0005)[11]+(0.0028)[12]+(-0.0050)[13]+(0.0022)[14]+(-0.0069)[15]
+(0.0053)[16]+(-0.0001)[17]+(-0.0074)[18]+(-0.0016)[19]

|[2651]= (0.0017)[1]+(-0.0046)[2]+(0.4916)[3]+(0.6381)[4]+(0.2992)[5]
+(0.2590)[6]+(0.3142)[7]+(0.3096)[8]+(0.0047)[9]+(0.0026)[10]
+(-0.0018)[11]+(0.0030)[12]+(0.0036)[13]+(-0.0029)[14]+(-0.0030)[15]
+(0.0013)[16]+(-0.0004)[17]+(0.0003)[18]+(-0.0051)[19]

|[3091]= (-0.7445)[1]+(-0.3342)[2]+(0.0028)[3]+(-0.0051)[4]+(-0.0271)[5]
+(0.0485)[6]+(0.0274)[7]+(-0.0431)[8]+(-0.0149)[9]+(-0.0013)[10]
+(0.0306)[11]+(0.0164)[12]+(0.0015)[13]+(-0.2910)[14]+(-0.1645)[15]
+(0.4637)[16]+(-0.0117)[17]+(-0.0058)[18]+(-0.0066)[19]

|[3109]= (-0.0729)[1]+(-0.0462)[2]+(-0.0584)[3]+(0.0888)[4]+(0.0827)[5]
+(-0.7525)[6]+(-0.1396)[7]+(0.5989)[8]+(0.1410)[9]+(0.0020)[10]
+(-0.0367)[11]+(0.0074)[12]+(-0.0261)[13]+(0.0128)[14]+(0.0687)[15]
+(0.0375)[16]+(-0.0141)[17]+(0.0410)[18]+(-0.0121)[19]

N 11/2-

|[2965]= (0.3283)[1]+(0.0026)[2]+(-0.9139)[3]+(0.1139)[4]+(0.1680)[5]
+(-0.0694)[6]+(0.1055)[7]

|[2980]= (-0.1205)[1]+(-0.0499)[2]+(-0.1887)[3]+(-0.9438)[4]+(0.0023)[5]
+(0.2114)[6]+(-0.1103)[7]

|[3004]= (-0.9339)[1]+(-0.0015)[2]+(-0.2805)[3]+(0.1573)[4]+(0.1194)[5]
+(-0.0285)[6]+(0.0976)[7]

|[3043]= (-0.0425)[1]+(-0.1824)[2]+(-0.1136)[3]+(0.1612)[4]+(-0.0181)[5]
+(0.0496)[6]+(-0.9609)[7]

N 13/2+

|[3116]= (0.7647)[1]+(0.3553)[2]+(-0.0295)[3]+(-0.0046)[4]+(0.3499)[5]
+(0.1301)[6]+(-0.3858)[7]+(-0.0031)[8]+(-0.0087)[9]

|[3147]= (0.4131)[1]+(0.1955)[2]+(0.0382)[3]+(0.0332)[4]+(-0.4157)[5]
+(-0.4228)[6]+(0.4721)[7]+(0.4630)[8]+(0.0003)[9]

|[3159]= (-0.2526)[1]+(-0.0968)[2]+(0.0201)[3]+(0.0117)[4]+(0.2803)[5]
+(-0.5037)[6]+(-0.5109)[7]+(0.5764)[8]+(-0.0300)[9]

|[3211]= (0.0548)[1]+(0.0856)[2]+(0.1926)[3]+(0.0683)[4]+(-0.3610)[5]
+(-0.5769)[6]+(-0.3462)[7]+(-0.6036)[8]+(0.0283)[9]

N 13/2-

|[3008]= (-0.9830)[1]+(0.1839)[2]

|[3141]= (-0.1839)[1]+(-0.9830)[2]

N 15/2+

|[3179]= (0.6129)[1]+(-0.7902)[2]

|[3256]= (0.7903)[1]+(0.6129)[2]

Table 10.3. Continued

Δ 1/2+

|1733⟩= (0.5743)| 1⟩+(0.5906)| 2⟩+(0.1753)| 3⟩+(0.3705)| 4⟩+(0.1169)| 5⟩
+(0.1399)| 6⟩+(0.0971)| 7⟩+(0.2037)| 8⟩+(-0.1949)| 9⟩+(0.0952)|10⟩
+(0.0111)|11⟩+(-0.1239)|12⟩+(0.0773)|13⟩+(-0.0290)|14⟩+(0.0100)|15⟩
+(0.0063)|16⟩

|1763⟩= (0.2039)| 1⟩+(0.2055)| 2⟩+(0.0704)| 3⟩+(0.1240)| 4⟩+(0.0441)| 5⟩
+(0.0549)| 6⟩+(0.0408)| 7⟩+(-0.5810)| 8⟩+(0.5581)| 9⟩+(-0.2571)|10⟩
+(0.0091)|11⟩+(0.3543)|12⟩+(-0.2037)|13⟩+(0.0851)|14⟩+(0.0047)|15⟩
+(0.0076)|16⟩

|2335⟩= (0.0073)| 1⟩+(-0.0362)| 2⟩+(0.0777)| 3⟩+(-0.0413)| 4⟩+(0.0028)| 5⟩
+(0.0246)| 6⟩+(0.0518)| 7⟩+(-0.0815)| 8⟩+(0.2866)| 9⟩+(0.7099)|10⟩
+(0.1460)|11⟩+(0.2537)|12⟩+(0.5419)|13⟩+(-0.0529)|14⟩+(0.0829)|15⟩
+(0.0780)|16⟩

|2430⟩= (0.0941)| 1⟩+(-0.1902)| 2⟩+(0.4499)| 3⟩+(-0.1997)| 4⟩+(-0.0500)| 5⟩
+(0.1005)| 6⟩+(0.3149)| 7⟩+(0.0431)| 8⟩+(-0.0624)| 9⟩+(-0.1561)|10⟩
+(0.5859)|11⟩+(-0.0511)|12⟩+(-0.1116)|13⟩+(0.0308)|14⟩+(0.3188)|15⟩
+(0.3319)|16⟩

Δ 1/2−

|1571⟩= (0.0200)| 1⟩+(0.0206)| 2⟩+(-0.0023)| 3⟩+(0.0022)| 4⟩+(-0.6379)| 5⟩
+(-0.5774)| 6⟩+(-0.2804)| 7⟩+(-0.3359)| 8⟩+(-0.1160)| 9⟩+(0.1683)|10⟩
+(-0.1611)|11⟩

|2151⟩= (0.7932)| 1⟩+(0.5690)| 2⟩+(-0.1872)| 3⟩+(0.0145)| 4⟩+(0.0001)| 5⟩
+(0.0597)| 6⟩+(-0.0622)| 7⟩+(0.0484)| 8⟩+(-0.0464)| 9⟩+(-0.0039)|10⟩
+(-0.0039)|11⟩

|2241⟩= (-0.0859)| 1⟩+(-0.0510)| 2⟩+(0.0389)| 3⟩+(-0.0022)| 4⟩+(-0.1089)| 5⟩
+(0.3878)| 6⟩+(-0.6832)| 7⟩+(0.3445)| 8⟩+(-0.4779)| 9⟩+(0.0755)|10⟩
+(-0.0833)|11⟩

|2452⟩= (-0.0041)| 1⟩+(-0.0053)| 2⟩+(0.0145)| 3⟩+(0.0124)| 4⟩+(-0.7353)| 5⟩
+(0.2990)| 6⟩+(0.2640)| 7⟩+(0.3239)| 8⟩+(0.1735)| 9⟩+(-0.2433)|10⟩
+(0.3253)|11⟩

Δ 3/2+

|1168⟩= (0.5493)| 1⟩+(0.5947)| 2⟩+(-0.0271)| 3⟩+(0.4578)| 4⟩+(0.1834)| 5⟩
+(-0.0353)| 6⟩+(-0.0079)| 7⟩+(0.2605)| 8⟩+(0.0815)| 9⟩+(0.1507)|10⟩
+(-0.0287)|11⟩+(-0.0087)|12⟩+(-0.0097)|13⟩+(-0.0042)|14⟩+(0.0002)|15⟩
+(-0.0177)|16⟩+(0.0033)|17⟩+(0.0206)|18⟩+(-0.0071)|19⟩+(0.0107)|20⟩
+(0.0030)|21⟩+(0.0032)|22⟩+(0.0178)|23⟩+(-0.0068)|24⟩+(0.0098)|25⟩
+(-0.0037)|26⟩+(-0.0038)|27⟩+(-0.0004)|28⟩

|1781⟩= (0.0130)| 1⟩+(-0.0436)| 2⟩+(-0.5808)| 3⟩+(-0.0527)| 4⟩+(-0.0332)| 5⟩
+(-0.5935)| 6⟩+(-0.1642)| 7⟩+(-0.0317)| 8⟩+(-0.0100)| 9⟩+(-0.0315)|10⟩
+(-0.3682)|11⟩+(-0.1204)|12⟩+(-0.1330)|13⟩+(-0.0852)|14⟩+(0.0027)|15⟩
+(-0.1934)|16⟩+(0.0061)|17⟩+(0.1680)|18⟩+(-0.0622)|19⟩+(0.1040)|20⟩
+(0.0062)|21⟩+(0.0036)|22⟩+(0.1059)|23⟩+(-0.0448)|24⟩+(0.0716)|25⟩
+(-0.0278)|26⟩+(-0.0336)|27⟩+(-0.0065)|28⟩

|1842⟩= (0.0061)| 1⟩+(-0.0111)| 2⟩+(0.2007)| 3⟩+(-0.0192)| 4⟩+(-0.0005)| 5⟩
+(0.1939)| 6⟩+(0.0732)| 7⟩+(-0.0149)| 8⟩+(-0.0065)| 9⟩+(-0.0006)|10⟩
+(0.1132)|11⟩+(0.0380)|12⟩+(0.0509)|13⟩+(0.0447)|14⟩+(0.0017)|15⟩
+(-0.5834)|16⟩+(-0.0038)|17⟩+(0.4532)|18⟩+(-0.2310)|19⟩+(0.3519)|20⟩
+(-0.0042)|21⟩+(-0.0023)|22⟩+(0.2565)|23⟩+(-0.1689)|24⟩+(0.2238)|25⟩
+(-0.1174)|26⟩+(-0.1276)|27⟩+(-0.0103)|28⟩

|1985⟩= (0.7631)| 1⟩+(-0.0962)| 2⟩+(0.0293)| 3⟩+(-0.4007)| 4⟩+(-0.2718)| 5⟩
+(0.0342)| 6⟩+(0.0169)| 7⟩+(-0.2867)| 8⟩+(-0.1064)| 9⟩+(-0.2737)|10⟩
+(0.0211)|11⟩+(0.0120)|12⟩+(0.0149)|13⟩+(0.0101)|14⟩+(-0.0001)|15⟩
+(0.0212)|16⟩+(0.0122)|17⟩+(-0.0046)|18⟩+(0.0190)|19⟩+(-0.0268)|20⟩
+(0.0054)|21⟩+(0.0081)|22⟩+(0.0037)|23⟩+(0.0144)|24⟩+(-0.0168)|25⟩
+(0.0118)|26⟩+(0.0114)|27⟩+(0.0004)|28⟩

Table 10.3. Continued

Δ 3/2-

[1605>= (-0.0039)[1>+(0.0242)[2>+(-0.0031)[3>+(0.0015)[4>+(0.0031)[5>
+(0.0208)[6>+(-0.0149)[7>+(0.6427)[8>+(0.5808)[9>+(0.2696)[10>
+(0.0080)[11>+(0.3368)[12>+(0.1065)[13>+(-0.1648)[14>+(0.1545)[15>
+(0.0060)[16>+(0.0052)[17>

[2166>= (0.7856)[1>+(-0.0551)[2>+(0.5621)[3>+(-0.1829)[4>+(0.0161)[5>
+(-0.0360)[6>+(0.0244)[7>+(-0.0011)[8>+(0.0470)[9>+(-0.0812)[10>
+(-0.0956)[11>+(0.0418)[12>+(-0.0534)[13>+(0.0157)[14>+(-0.0118)[15>
+(-0.0686)[16>+(-0.0887)[17>

[2233>= (-0.0884)[1>+(-0.7665)[2>+(-0.0610)[3>+(0.0192)[4>+(-0.0060)[5>
+(-0.4295)[6>+(0.3950)[7>+(-0.0250)[8>+(0.1002)[9>+(-0.1065)[10>
+(-0.1246)[11>+(0.0884)[12>+(-0.0796)[13>+(-0.0075)[14>+(-0.0031)[15>
+(-0.0877)[16>+(-0.0883)[17>

[2278>= (-0.1069)[1>+(0.1914)[2>+(-0.0654)[3>+(0.0334)[4>+(-0.0078)[5>
+(0.1005)[6>+(-0.1034)[7>+(-0.1364)[8>+(0.3580)[9>+(-0.6190)[10>
+(-0.2542)[11>+(0.3112)[12>+(-0.4374)[13>+(0.0413)[14>+(-0.0662)[15>
+(-0.1677)[16>+(-0.0951)[17>

Δ 5/2+

[1854>= (-0.5955)[1>+(-0.5969)[2>+(-0.1688)[3>+(0.0118)[4>+(-0.3615)[5>
+(-0.1220)[6>+(-0.1322)[7>+(-0.0871)[8>+(0.0010)[9>+(0.0104)[10>
+(0.0066)[11>+(0.0031)[12>+(-0.1867)[13>+(0.1493)[14>+(-0.0669)[15>
+(0.1060)[16>+(-0.0080)[17>+(0.0873)[18>+(-0.0484)[19>+(0.0675)[20>
+(-0.0309)[21>+(-0.0381)[22>+(-0.0044)[23>+(-0.0094)[24>+(-0.0027)[25>
+(-0.0014)[26>

[1883>= (0.1902)[1>+(0.1859)[2>+(0.0521)[3>+(0.0031)[4>+(0.1093)[5>
+(0.0396)[6>+(0.0396)[7>+(0.0257)[8>+(-0.0014)[9>+(0.0022)[10>
+(0.0035)[11>+(0.0010)[12>+(-0.5937)[13>+(0.4374)[14>+(-0.2302)[15>
+(0.3772)[16>+(0.0054)[17>+(0.2327)[18>+(-0.1630)[19>+(0.2345)[20>
+(-0.1274)[21>+(-0.1408)[22>+(-0.0099)[23>+(0.0037)[24>+(0.0045)[25>
+(-0.0009)[26>

[2400>= (0.0171)[1>+(-0.0238)[2>+(0.0499)[3>+(-0.1027)[4>+(-0.0242)[5>
+(-0.0073)[6>+(0.0070)[7>+(0.0354)[8>+(0.0082)[9>+(-0.0688)[10>
+(-0.0362)[11>+(-0.0221)[12>+(0.0822)[13>+(-0.0293)[14>+(-0.7028)[15>
+(-0.3347)[16>+(-0.0642)[17>+(-0.0024)[18>+(-0.5117)[19>+(-0.2185)[20>
+(-0.1459)[21>+(0.1620)[22>+(-0.0221)[23>+(-0.0375)[24>+(-0.0316)[25>
+(0.0070)[26>

[2419>= (0.0028)[1>+(-0.0163)[2>+(0.0011)[3>+(-0.7837)[4>+(-0.0139)[5>
+(-0.0029)[6>+(-0.0078)[7>+(0.0027)[8>+(-0.0017)[9>+(-0.5510)[10>
+(-0.1659)[11>+(-0.1778)[12>+(-0.0082)[13>+(0.0290)[14>+(0.0938)[15>
+(0.0317)[16>+(0.0599)[17>+(0.0204)[18>+(0.0634)[19>+(0.0321)[20>
+(0.0250)[21>+(-0.0110)[22>+(0.0071)[23>+(0.0406)[24>+(0.0215)[25>
+(-0.0069)[26>

Δ 5/2-

[2212>= (0.7997)[1>+(-0.0613)[2>+(0.5517)[3>+(-0.2118)[4>+(-0.0001)[5>
+(-0.0397)[6>+(0.0263)[7>+(-0.0007)[8>+(-0.0426)[9>+(-0.0400)[10>
+(-0.0236)[11>+(-0.0201)[12>+(-0.0298)[13>+(-0.0106)[14>+(0.0072)[15>
+(-0.0060)[16>

[2262>= (0.0677)[1>+(0.4593)[2>+(0.0457)[3>+(-0.0181)[4>+(0.0135)[5>
+(0.2457)[6>+(-0.2423)[7>+(0.0095)[8>+(-0.0330)[9>+(0.6518)[10>
+(-0.0220)[11>+(-0.0103)[12>+(0.4290)[13>+(0.0079)[14>+(-0.1868)[15>
+(0.1329)[16>

[2299>= (-0.0288)[1>+(-0.6531)[2>+(-0.0217)[3>+(0.0007)[4>+(-0.0011)[5>
+(-0.3482)[6>+(0.3407)[7>+(-0.0044)[8>+(-0.0207)[9>+(0.4664)[10>
+(-0.0119)[11>+(-0.0078)[12>+(0.2988)[13>+(-0.0129)[14>+(-0.1290)[15>
+(0.1040)[16>

Table 10.3. Continued

|2368>= (0.0471)| 1>+(-0.0012)| 2>+(0.0228)| 3>+(-0.0196)| 4>+(-0.0118)| 5>
+(0.0036)| 6>+(0.0028)| 7>+(-0.0024)| 8>+(0.8004)| 9>+(0.0379)|10>
+(0.5067)|11>+(0.3121)|12>+(0.0182)|13>+(-0.0104)|14>+(-0.0175)|15>
+(0.0073)|16>

Δ 7/2+

|1929>= (0.6373)| 1>+(0.6143)| 2>+(0.2012)| 3>+(-0.0160)| 4>+(0.3560)| 5>
+(0.1229)| 6>+(0.1432)| 7>+(0.1122)| 8>+(0.0022)| 9>+(-0.0142)|10>
+(-0.0078)|11>+(-0.0040)|12>+(-0.0143)|13>+(-0.0029)|14>+(-0.0134)|15>
+(-0.0107)|16>+(-0.0094)|17>+(0.0031)|18>+(-0.0014)|19>+(-0.0103)|20>
+(-0.0071)|21>+(-0.0073)|22>+(0.0017)|23>

|2450>= (-0.0061)| 1>+(0.0375)| 2>+(-0.0199)| 3>+(0.7724)| 4>+(0.0319)| 5>
+(0.0079)| 6>+(0.0090)| 7>+(-0.0165)| 8>+(-0.0061)| 9>+(0.5415)|10>
+(0.1530)|11>+(0.1683)|12>+(0.0206)|13>+(0.0695)|14>+(0.1697)|15>
+(0.0144)|16>+(0.0087)|17>+(-0.0060)|18>+(0.0368)|19>+(0.0794)|20>
+(0.0793)|21>+(0.0834)|22>+(-0.0024)|23>

|2494>= (-0.0026)| 1>+(0.0036)| 2>+(-0.0124)| 3>+(-0.0074)| 4>+(0.0019)| 5>
+(0.0015)| 6>+(0.0012)| 7>+(-0.0111)| 8>+(0.0040)| 9>+(-0.0109)|10>
+(0.0025)|11>+(0.0064)|12>+(0.0380)|13>+(0.7439)|14>+(-0.2831)|15>
+(0.0234)|16>+(0.0187)|17>+(-0.0031)|18>+(0.4815)|19>+(-0.2336)|20>
+(0.2308)|21>+(-0.1553)|22>+(0.0075)|23>

|2527>= (0.0373)| 1>+(-0.0698)| 2>+(0.0979)| 3>+(0.1904)| 4>+(-0.0709)| 5>
+(-0.0100)| 6>+(0.0102)| 7>+(0.0660)| 8>+(0.0115)| 9>+(0.1344)|10>
+(0.0255)|11>+(0.0435)|12>+(-0.1729)|13>+(-0.2471)|14>+(-0.7002)|15>
+(-0.1058)|16>+(-0.0787)|17>+(0.0060)|18>+(-0.1525)|19>+(-0.3349)|20>
+(-0.2507)|21>+(-0.3478)|22>+(0.0060)|23>

Δ 7/2-

|2307>= (0.2492)| 1>+(0.0048)| 2>+(0.1291)| 3>+(-0.1307)| 4>+(-0.0091)| 5>
+(0.0084)| 6>+(0.7669)| 7>+(0.4923)| 8>+(-0.0171)| 9>+(-0.2172)|10>
+(0.1618)|11>+(0.0085)|12>

|2352>= (-0.7676)| 1>+(-0.0128)| 2>+(-0.3858)| 3>+(0.4072)| 4>+(-0.0077)| 5>
+(0.0033)| 6>+(0.2518)| 7>+(0.1563)| 8>+(-0.0230)| 9>+(-0.0721)|10>
+(0.0499)|11>+(-0.0039)|12>

|2905>= (-0.0946)| 1>+(-0.0154)| 2>+(0.8027)| 3>+(0.5842)| 4>+(-0.0371)| 5>
+(-0.0140)| 6>+(-0.0079)| 7>+(0.0251)| 8>+(0.0123)| 9>+(0.0213)|10>
+(-0.0305)|11>+(-0.0393)|12>

|2935>= (0.0058)| 1>+(0.0291)| 2>+(-0.0369)| 3>+(-0.0128)| 4>+(0.0084)| 5>
+(-0.0464)| 6>+(-0.0871)| 7>+(0.5455)| 8>+(0.0974)| 9>+(0.8146)|10>
+(-0.1304)|11>+(-0.0178)|12>

Δ 9/2+

|2492>= (-0.7887)| 1>+(0.0091)| 2>+(-0.5486)| 3>+(-0.1406)| 4>+(-0.1713)| 5>
+(0.0085)| 6>+(-0.0035)| 7>+(-0.0974)| 8>+(-0.0869)| 9>+(-0.0575)|10>
+(-0.0328)|11>+(-0.0677)|12>+(-0.0415)|13>+(0.0063)|14>+(0.0028)|15>
+(-0.0065)|16>

|2508>= (-0.0481)| 1>+(-0.0017)| 2>+(-0.0345)| 3>+(-0.0087)| 4>+(-0.0120)| 5>
+(0.0084)| 6>+(-0.0069)| 7>+(0.7084)| 8>+(-0.3669)| 9>+(0.4498)|10>
+(-0.2706)|11>+(0.2120)|12>+(-0.1986)|13>+(0.0062)|14>+(0.0057)|15>
+(-0.0011)|16>

|2591>= (0.1243)| 1>+(0.0020)| 2>+(0.0891)| 3>+(0.0105)| 4>+(0.0224)| 5>
+(0.0030)| 6>+(-0.0076)| 7>+(-0.3524)| 8>+(-0.7083)| 9>+(-0.2161)|10>
+(-0.3113)|11>+(-0.2786)|12>+(-0.3589)|13>+(-0.0143)|14>+(0.0036)|15>
+(0.0052)|16>

|3086>= (-0.0198)| 1>+(-0.0279)| 2>+(0.0081)| 3>+(0.1204)| 4>+(-0.0493)| 5>
+(0.1934)| 6>+(0.0452)| 7>+(0.0555)| 8>+(-0.0866)| 9>+(-0.0089)|10>
+(0.7480)|11>+(0.0826)|12>+(-0.5892)|13>+(-0.1309)|14>+(0.0268)|15>
+(0.0258)|16>

Table 10.3. Continued

Δ 9/2-

[2409>= (0.8139)[1>+(0.3855)[2>+(-0.4347)[3>+(-0.0072)[4>+(-0.0003)[5>
+(0.0063)[6>+(0.0004)[7>+(-0.0110)[8>

[2928>= (0.0822)[1>+(-0.8160)[2>+(-0.5691)[3>+(0.0092)[4>+(0.0258)[5>
+(0.0274)[6>+(0.0384)[7>+(-0.0272)[8>

[2959>= (0.0063)[1>+(-0.0389)[2>+(-0.0313)[3>+(0.0430)[4>+(-0.3233)[5>
+(0.0217)[6>+(-0.8938)[7>+(0.3032)[8>

[2988>= (-0.0009)[1>+(0.0086)[2>+(0.0042)[3>+(-0.0449)[4>+(0.9430)[5>
+(-0.0180)[6>+(-0.3243)[7>+(0.0582)[8>

Δ 11/2+

[2535>= (-0.8043)[1>+(-0.5496)[2>+(-0.1299)[3>+(-0.1845)[4>+(0.0106)[5>
+(-0.0033)[6>+(0.0084)[7>+(0.0079)[8>+(0.0043)[9>+(0.0062)[10>

[3115>= (0.0022)[1>+(-0.0204)[2>+(-0.0225)[3>+(0.0135)[4>+(-0.9161)[5>
+(-0.3596)[6>+(-0.0131)[7>+(-0.0044)[8>+(-0.0786)[9>+(-0.1556)[10>

[3143>= (-0.0011)[1>+(-0.0001)[2>+(0.0079)[3>+(-0.0047)[4>+(0.0698)[5>
+(-0.0041)[6>+(0.0185)[7>+(0.0112)[8>+(0.6675)[9>+(-0.7410)[10>

[3172>= (-0.1474)[1>+(0.1094)[2>+(0.9189)[3>+(-0.3236)[4>+(-0.0296)[5>
+(-0.0516)[6>+(0.0423)[7>+(0.0325)[8>+(0.0726)[9>+(0.0762)[10>

Δ 11/2-

[2984>= (0.0250)[1>+(-0.2485)[2>+(-0.9514)[3>+(0.1805)[4>

[3012>= (0.0328)[1>+(-0.9669)[2>+(0.2531)[3>+(-0.0019)[4>

[3105>= (-0.1676)[1>+(-0.0497)[2>+(-0.1752)[3>+(-0.9690)[4>

[3144>= (0.9851)[1>+(0.0301)[2>+(-0.0141)[3>+(-0.1694)[4>

Δ 13/2+

[3138>= (0.9094)[1>+(0.3943)[2>+(0.0917)[3>+(0.0961)[4>

[3151>= (-0.0090)[1>+(0.0209)[2>+(-0.7213)[3>+(0.6923)[4>

[3218>= (-0.0977)[1>+(-0.1078)[2>+(0.6842)[3>+(0.7148)[4>

[3291>= (-0.4042)[1>+(0.9125)[2>+(0.0576)[3>+(0.0273)[4>

Δ 13/2-

[3037>= (1.0000)[1>

Δ 15/2+

[3163>= (0.9064)[1>+(0.4225)[2>

[3329>= (-0.4225)[1>+(0.9064)[2>

D. Table of the Eigenvector Characteristics

Table 10.4 is a listing of the major percentages of basis functions in Table 10.2, with particular L, S, and j values, contained in the eigenvectors of Table 10.3. The L and S values are the total orbital and spin angular momentum, respectively. The j value is the S_3 irrep in the space variables.

Table 10.4. The major percentages of basis functions, with particular L, S, and j values, contained in the eigenvectors

	Eigenvalue	L	S	j	Basis Function Percentage
N 1/2+	954	0	1/2	[3]	99%
	1583	2	3/2	[2,1]	64%
		0	1/2	[2,1]	20%
	1689	0	1/2	[2,1]	70%
		2	3/2	[2,1]	27%
	1822	1	1/2	$[1^3]$	82%
N 1/2-	1357	1	3/2	[2,1]	75%
		1	1/2	[2,1]	25%
	1486	1	1/2	[2,1]	74%
		1	3/2	[2,1]	25%
	2080	1	1/2	[3]	84%
	2149	1	1/2	$[1^3]$	69%
		1	3/2	[2,1]	21%
N 3/2+	1676	2	1/2	[2,1]	42%
		2	3/2	[2,1]	26%
	1725	2	1/2	[3]	44%
		2	3/2	[2,1]	50%
	1757	0	3/2	[2,1]	73%
	1823	2	1/2	[3]	34%
		2	1/2	[2,1]	41%
N 3/2-	1531	1	1/2	[2,1]	48%
		1	3/2	[2,1]	51%
	1577	1	1/2	[2,1]	51%
		1	3/2	[2,1]	49%
	2113	1	1/2	[3]	91%
	2183	1	1/2	$[1^3]$	22%
		3	3/2	[2,1]	38%
N 5/2+	1794	2	1/2	[3]	71%
		2	1/2	[2,1]	27%
	1845	2	3/2	[2,1]	74%
	1907	2	1/2	[2,1]	61%
		2	3/2	[2,1]	24%
	2353	2	1/2	[2,1]	85%
N 5/2-	1685	1	3/2	[2,1]	100%
	2202	3	1/2	[2,1]	52%
	2228	3	1/2	[3]	19%
		3	1/2	[2,1]	19%
		3	3/2	[2,1]	34%
	2267	3	3/2	[2,1]	52%
		1	3/2	[2,1]	23%

Table 10.4. Continued

	Eigenvalue	L	S	j	Basis Function Percentage
N 7/2+	1982	2	3/2	[2,1]	100%
	2403	4	1/2	[3]	44%
		4	1/2	[2,1]	37%
	2428	4	1/2	[3]	31%
		4	3/2	[2,1]	30%
	2442	2	3/2	[2,1]	82%
N 7/2-	2275	3	1/2	[2,1]	90%
	2296	3	3/2	[2,1]	92%
	2346	3	1/2	[3]	84%
	2367	3	1/2	$[1^3]$	76%
N 9/2+	2452	4	1/2	[3]	68%
		4	1/2	[2,1]	29%
	2500	4	3/2	[2,1]	54%
		4	1/2	[3]	20%
		4	1/2	[2,1]	24%
	2527	4	1/2	[2,1]	48%
		4	3/2	[2,1]	41%
	2574	4	3/2	[2,1]	83%
N 9/2-	2351	3	3/2	[2,1]	100%
	2932	5	1/2	[2,1]	77%
		5	1/2	[3]	20%
	2951	5	3/2	[2,1]	82%
	2971	3	3/2	[2,1]	87%
N 11/2+	2563	4	3/2	[2,1]	100%
	2651	4	3/2	[2,1]	100%
	3091	6	1/2	[3]	67%
		6	1/2	[2,1]	30%
	3109	4	3/2	[2,1]	98%
N 11/2-	2965	5	1/2	[2,1]	86%
	2980	5	3/2	[2,1]	94%
	3004	5	1/2	[3]	87%
	3034	5	1/2	$[1^3]$	92%

Table 10.4. Continued

	Eigenvalue	L	θ	j	Basis Function Percentage
N 13/2+	3116	6	1/2	[3]	71%
		6	1/2	[2,1]	27%
	3147	6	1/2	[2,1]	40%
		6	3/2	[2,1]	39%
		6	1/2	[3]	21%
	3159	6	1/2	[2,1]	34%
		6	3/2	[2,1]	59%
	3211	6	3/2	[2,1]	70%
		6	1/2	[2,1]	25%
N 13/2-	3008	5	3/2	[2,1]	100%
	3141	5	3/2	[2,1]	100%
N 15/2+	3179	6	3/2	[2,1]	100%
	3256	6	3/2	[2,1]	100%
Δ 1/2+	1733	2	3/2	[3]	89%
	1763	0	1/2	[2,1]	82%
	2335	0	1/2	[2,1]	45%
		1	1/2	[2,1]	54%
	2430	2	3/2	[3]	40%
		1	1/2	[2,1]	58%
Δ 1/2-	1571	1	1/2	[2,1]	100%
	2151	1	3/2	[3]	99%
	2241	1	1/2	[2,1]	99%
	2452	1	1/2	[2,1]	100%
Δ 3/2+	1168	0	3/2	[3]	100%
	1781	2	3/2	[3]	89%
	1842	2	1/2	[2,1]	90%
	1985	0	3/2	[3]	99%
Δ 3/2-	1605	1	1/2	[2,1]	100%
	2166	1	3/2	[3]	97%
	2233	3	3/2	[3]	93%
	2278	1	1/2	[2,1]	82%
Δ 5/2+	1854	2	3/2	[3]	91%
	1883	2	1/2	[2,1]	91%
	2400	2	1/2	[2,1]	97%
	2419	4	3/2	[3]	98%

Table 10.4. Continued

	Eigenvalue	L	S	j	Basis Function Percentage
Δ 5/2-	2212	1	3/2	[3]	99%
	2262	3	1/2	[2,1]	66%
		3	3/2	[3]	33%
	2299	3	3/2	[3]	66%
		3	1/2	[2,1]	33%
	2368	2	1/2	[2,1]	99%
Δ 7/2+	1929	2	3/2	[3]	100%
	2450	4	3/2	[3]	94%
	2494	4	1/2	[2,1]	100%
	2527	4	1/2	[2,1]	87%
Δ 7/2-	2307	3	1/2	[2,1]	90%
	2352	3	3/2	[3]	90%
	2905	3	3/2	[3]	99%
	2935	3	1/2	[2,1]	100%
Δ 9/2+	2492	4	3/2	[3]	97%
	2508	4	1/2	[2,1]	100%
	2591	4	1/2	[2,1]	98%
	3086	4	1/2	[2,1]	94%
Δ 9/2-	2409	3	3/2	[3]	100%
	2928	3	3/2	[3]	100%
	2959	5	1/2	[2,1]	89%
	2988	5	3/2	[3]	89%
Δ 11/2+	2535	4	3/2	[3]	100%
	3115	6	3/2	[3]	97%
	3143	6	1/2	[2,1]	99%
	3172	4	3/2	[3]	98%
Δ 11/2-	2984	5	1/2	[2,1]	94%
	3012	5	3/2	[3]	93%
	3105	5	1/2	[2,1]	97%
	3144	4	3/2	[3]	97%

Table 10.4. Continued

	Eigenvalue	L	S	j	Basis Function Percentage
Δ 13/2+	3138	6	3/2	[3]	98%
	3151	6	1/2	[2,1]	100%
	3218	6	1/2	[2,1]	98%
	3291	6	3/2	[3]	100%
Δ 13/2-	3037	5	3/2	[3]	100%
Δ 15/2+	3163	6	3/2	[3]	100%
	3329	6	3/2	[3]	100%

XI. APPENDIX E: THE FOURIER TRANSFORM OF A SINGLE PARTICLE, HARMONIC OSCILLATOR WAVEFUNCTION AND MATRIX ELEMENTS OF MOMENTUM OPERATORS

The Fourier transform to momentum space, of a wavefunction in coordinate space, is useful in calculating matrix elements of momentum operators. The Fourier transform, $\Phi(\underset{\sim}{p})$, of wavefunction in coordinate space, $\Psi(\underset{\sim}{r})$, is defined as

$$\Phi(\underset{\sim}{p}) = (2\pi)^{-3/2} \int d^3r\, e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}} \Psi(\underset{\sim}{r}) \quad . \tag{11.1}$$

In (11.1), $\hbar$ has been set equal to one, $\underset{\sim}{p}$ is the momentum variable, and the integral is over all of $\underset{\sim}{r}$ space. The inverse Fourier transform is

$$\Psi(\underset{\sim}{r}) = (2\pi)^{-3/2} \int d^3p\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}} \Phi(\underset{\sim}{p}) \tag{11.2}$$

and the Dirac delta function is defined as

$$\delta^3(\underset{\sim}{r} - \underset{\sim}{r}') = (2\pi)^{-3} \int d^3p\, e^{i\underset{\sim}{p}\cdot(\underset{\sim}{r}-\underset{\sim}{r}')} \quad . \tag{11.3}$$

For ease in performing calculations, one may take the vectors $\underset{\sim}{r}$ and $\underset{\sim}{p}$ to be dimensionless. Their components satisfy the commutation relation

$$[r_k, p_j] = i\delta_{jk} \quad .$$

This transformation theory allows $\underset{\sim}{p}$ and $\underset{\sim}{r}$ equal footing as dynamical variables in quantum mechanics.

In terms of these dimensionless vectors $\underset{\sim}{r}$ and $\underset{\sim}{p}$, the single particle, harmonic oscillator Hamiltonian is

$$H = \frac{\omega}{2}(\underset{\sim}{p}^2 + \underset{\sim}{r}^2) \tag{11.4}$$

where ω is the oscillator frequency. In spherical coordinates, $\underset{\sim}{r} = (r,\theta,\phi)$, the wavefunction $\Psi(\underset{\sim}{r})$ which satisfies the Schrodinger equation

$$H\Psi_{n\ell m}(\underset{\sim}{r}) = E_{n\ell m}\Psi(\underset{\sim}{r})$$

is

$$\Psi_{n\ell m}(\underset{\sim}{r}) = N_{n\ell}\, r^{\ell}\, L_n^{\ell+\frac{1}{2}}(r^2)\, e^{-\frac{1}{2}r^2}\, Y_{\ell m}(\hat{r}) \quad ,$$

$$N_{n\ell} = \left(\frac{2n!}{\Gamma(n+\ell+3/2)}\right)^{1/2} \quad . \tag{11.5}$$

$L_n^{\ell+\frac{1}{2}}(x)$ is an associated Laguerre polynomial (20) and $Y_{\ell m}(\hat{r})$ is a spherical harmonic function (25) where $\hat{r} = \underset{\sim}{r}/r$. The labels n, ℓ, and m are the radial, angular momentum, and angular momentum projection quantum numbers, respectively. $E_{n\ell m}$, n, ℓ, and m may take the values

$$E_{n\ell m} = \omega(2n + \ell + 3/2) \quad ,$$

$$n = 0, 1, 2, \ldots \quad ,$$

$$\ell = 0, 1, 2, \ldots \quad ,$$

and

$$m = -\ell, -\ell+1, \ldots, \ell \quad .$$

To find the Fourier transform of $\Psi_{n\ell m}(\underset{\sim}{r})$, substitute (11.5) into (11.1) to give the explicit formula

$$\Phi_{n\ell m}(\underset{\sim}{p}) = (2\pi)^{-3/2} \int d^3r\, e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}}\, N_{n\ell}\, r^{\ell}\, L_n^{\ell+\frac{1}{2}}(r^2)\, e^{-\frac{1}{2}r^2}\, Y_{\ell m}(\hat{r}) \quad . \tag{11.6}$$

The next step is to use the expansion of $e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}}$ in terms of spherical harmonics (23)

$$e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}} = 4\pi \sum_{L=0}^{\infty} \sum_{M=-L}^{L} (-i)^L\, j_L(pr)\, Y_{LM}(\hat{p})\, Y^*_{LM}(\hat{r}) \tag{11.7}$$

where j_L is the spherical Bessel function of order L (20). Inserting (11.7) into (11.6) and integrating term by term over the solid angle Ω gives

$$\Phi_{n\ell m}(\underset{\sim}{p}) = N_{n\ell} \left(\frac{2}{\pi}\right)^{1/2} (-i)^{\ell}\, Y_{\ell m}(\hat{p}) \left[\int_0^{\infty} r^2 dr\, j_{\ell}(pr)\, r^{\ell}\, L^{\ell+\frac{1}{2}}(r^2)\, e^{-\frac{1}{2}r^2}\right] \tag{11.8}$$

where the orthonormality of the spherical harmonics gives

$$\int d\Omega\, Y^*_{LM}(\hat{r})\, Y_{\ell m}(\hat{r}) = \delta_{L\ell}\delta_{Mm} \quad .$$

Now the integration in (11.8) must be performed. Writing j_{ℓ} as a Bessel function of half-odd order,

$$j_{\ell}(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{\ell+\frac{1}{2}}(x) \quad ,$$

and substituting x for r^2 gives

$$\Phi_{n\ell m}(\underset{\sim}{p}) = \frac{1}{2} N_{n\ell} (-i)^{\ell}\, Y_{\ell m}(\hat{p})\, p^{-1/2}$$

$$\left[\int_0^\infty dx\, x^{\frac{1}{2}(\ell+\frac{1}{2})}\, J_{\ell+\frac{1}{2}}(p\sqrt{x})\, L_n^{\ell+\frac{1}{2}}(x)\, e^{-\frac{1}{2}x} \right] \quad . \qquad (11.9)$$

The integral in the bracket is equal to

$$2(-1)^n\, e^{-\frac{1}{2}p^2}\, p^{\ell+\frac{1}{2}}\, L_n^{\ell+\frac{1}{2}}(p^2)$$

which may be found by using a formula given by Lebedev (33). This gives the result

$$\Phi_{n\ell m}(\underset{\sim}{p}) = (-1)^{n+\ell}(-i)^{\ell}[N_{n\ell}\, p^{\ell}\, L_n^{\ell+\frac{1}{2}}(p^2)\, e^{-\frac{1}{2}p^2}\, Y_{\ell m}(\hat{p})] \quad .$$

The quantity in brackets is, by (11.5), $\Psi_{n\ell m}(\underset{\sim}{p})$. Thus, the Fourier transform of a single particle, harmonic oscillator wavefunction in coordinate space is the same wavefunction in momentum space multiplied by a phase factor. That is,

$$\begin{aligned} \Phi_{n\ell m}(\underset{\sim}{p}) &= (2\pi)^{-3/2} \int d^3r\, e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}}\, \Psi_{n\ell m}(\underset{\sim}{r}) \\ &= (-1)^{n+\ell}(i)^{\ell}\, \Psi_{n\ell m}(\underset{\sim}{p}) \quad . \end{aligned} \qquad (11.10)$$

The result is not surprising, given the form of the Hamiltonian in Eq. (11.4). There $\underset{\sim}{p}$ and $\underset{\sim}{r}$ occur with identical form and, hence, a wavefunction in the $\underset{\sim}{r}$ variable should have the same form, to within an overall phase factor, as the transformed wavefunction in the $\underset{\sim}{p}$ variable.

This formula can be used to simplify matrix elements of momentum operators between harmonic oscillator wavefunctions in coordinate space.

Since $\underset{\sim}{p} = -i\underset{\sim}{\nabla}$ in coordinate space, the process of taking derivatives will be avoided. Suppose that one has the matrix element of an irreducible tensor operator in the $\underset{\sim}{p}$ variable, $T_{LM}(\underset{\sim}{p})$ (see (7.20) for details of T_{LM}),

$$\langle n'\ell'm'|T_{LM}(\underset{\sim}{p})|n\ell m\rangle = \int d^3r\, \psi^*_{n'\ell'm'}(\underset{\sim}{r})\, T_{LM}(\underset{\sim}{p})\, \psi_{n\ell m}(\underset{\sim}{r}) \quad . \quad (11.11)$$

Substituting (11.3) and (11.10) for the wavefunctions in (11.11) gives

$$\langle n'\ell'm'|T_{LM}(\underset{\sim}{p})|n\ell m\rangle = (-1)^{n+n'+\ell}(i)^{\ell+\ell'}(2\pi)^{-3} \int\int\int d^3r\, d^3k\, d^3k' \times [e^{-i\underset{\sim}{k}'\cdot\underset{\sim}{r}}\, \psi^*_{n'\ell'm'}(\underset{\sim}{k}')\, T_{LM}(-i\underset{\sim}{\nabla})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, \psi_{n\ell m}(\underset{\sim}{k})] \quad . \quad (11.12)$$

Operating to the right with T_{LM} gives

$$T_{LM}(-i\underset{\sim}{\nabla})e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = T_{LM}(\underset{\sim}{k})e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$$

and so by using the Dirac delta function of (11.4) and integrating, (11.12) becomes

$$\langle n'\ell'm'|T_{LM}(\underset{\sim}{p})|n\ell m\rangle = (-1)^{n+n'+\ell}(i)^{\ell+\ell'} \int d^3k\, \psi^*_{n'\ell'm'}(\underset{\sim}{k})\, T_{LM}(\underset{\sim}{k})\, \psi_{n\ell m}(\underset{\sim}{k}).$$

Comparison with (11.11) shows that

$$\langle n'\ell'm'|T_{LM}(\underset{\sim}{p})|n\ell m\rangle = (-1)^{n+n'+\ell}(i)^{\ell+\ell'}\langle n'\ell'm'|T_{LM}(\underset{\sim}{r})|n\ell m\rangle \quad . \quad (11.13)$$

Therefore, the matrix element of a momentum operator is the same, to within an overall phase factor, as the matrix element with the momentum variable replaced by the coordinate variable $\underset{\sim}{r}$. Thus, the complicated process of taking derivatives has been avoided.

XII. BIBLIOGRAPHY

1. R. W. Berard et al., Phys. Lett. 47B, 355 (1973).

2. N. W. Dean, Introduction to the Strong Interactions (Gordan and Breach, New York, 1976).

3. J. J. J. Kokkedee, The Quark Model (Benjamin, New York, 1969).

4. R. L. Kelly et al., Rev. Mod. Phys. 52, S1 (1980).

5. S. A. Williams, D. L. Pursey, P. D. Morley, and F. J. Margetan, Phys. Rev. Lett. 49, 771 (1982).

6. A. DeRujula, H. Georgi, and S. L. Glashow, Phys. Rev. D 12, 147 (1975).

7. V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory (Pergamon Press, New York, 1971), p. 83.

8. G. Breit, Phys. Rev. 34, 553 (1929).

9. N. Isgur and G. Karl, Phys. Rev. D 18, 4187 (1978).

10. N. Isgur and G. Karl, Phys. Rev. D 19, 2653 (1979).

11. N. Isgur and G. Karl, Phys. Rev. D 20, 1191 (1979).

12. R. Koniuk and N. Isgur, Phys. Rev. D 21, 1868 (1980).

13. N. Isgur and G. Karl, Phys. Rev. D 21, 3175 (1980).

14. F. Borkowski, P. Peuser, G. G. Simon, V. H. Walther, and R. D. Wendling, Nucl. Phys. A222, 269 (1974).

15. W. Celmaster, Phys. Rev. D 15, 1391 (1977).

16. D. Gromes and I. O. Stamatescu, Nucl. Phys. B112, 213 (1976).

17. D. Gromes, Nucl. Phys. B130, 18 (1977).

18. T. A. DeGrand and R. L. Jaffe, Ann. Phys. 100, 425 (1976).

19. T. A. DeGrand, Ann. Phys. 101, 496 (1976).

20. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55 (U.S. Government Printing Office, Washington, DC, 1972).

21. J. D. Jackson, Classical Electrodynamics (John Wiley and Sons, New York, 1975), p. 111.

22. N. Isgur, G. Karl, and R. Koniuk, Phys. Rev. Lett. 41, 1269 (1978).

23. E. Merzbacher, Quantum Mechanics (John Wiley and Sons, New York, 1970), pp. 189 and 198.

24. B. C. Carlson, Special Functions of Applied Mathematics (Academic Press, New York, 1977), p. 16.

25. M. E. Rose, Elementary Angular Momentum Theory (John Wiley and Sons, New York, 1957).

26. M. Hamermesh, Group Theory (Addison-Wesley, Reading, MA, 1962).

27. M. Baranger and K. T. R. Davies, Nucl. Phys. 79, 403 (1966).

28. J. D. Talman, Nucl. Phys. A141, 273 (1970).

29. B. S. Cooper, USAEC Ames Laboratory Contribution No. 2702.

30. I. Talmi, Helv. Phys. Acta 25, 185 (1952).

31. M. Moshinsky, Nucl. Phys. 13, 104 (1959).

32. A. P. Yutsis, I. B. Levinson, and V. V. Vanagas, Mathematical Apparatus of the Theory of Angular Momentum (Acad. Sci. Lith. S. S. R., Publication No. 3, Vilnius, 1960), Eq. (26.4).

33. N. N. Lebedev, Special Functions and Their Applications (Dover, New York, 1972), p. 83.

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